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HYDRODYNAMIC STABILITY ANALYSIS FOR VARIABLE VISCOUS FLUID FLOW THROUGH A POROUS MEDIUM

Samuel O. Adesanya[1], J.A. Falade[2]
[1]Department of Mathematical Sciences
Redeemer's University
NIGERIA
[2]Department of Physical Sciences
Redeemer's University
NIGERIA

Abstract: This paper investigates the effect of variable viscosity on the temporal development of small disturbances in a pressure – driven fluid flow through a channel saturated with porous medium. By assuming a periodic solution of the Squire form, a linearized fourth-order eigenvalue problem is obtained and solved using Adomian decomposition method (ADM). The results of the computation showed that increase in viscosity variation parameter stabilizes the flow while increase in porous permeability parameter has a destabilizing effect on the flow.

Key Words: ADM, variable viscosity, porous medium, linear stability analysis

1. Introduction

Studies related to hydrodynamic stability of a moving viscous fluid through a channel filled with saturated porous medium have been of great importance in the last few years due to its important applications in transport processes, petro-chemical engineering and geo-physical flows. This is due to the fact that

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the study provides valuable information on the progression of fluids from lamina to turbulent flows. It is well known that turbulent flows may have undesirable effect in some real life applications. For instance, in arterial blood flow with multiple stenosis, shipping over deep seas and in aviation industry and many more.

In recent times, several work has been done in this area of research for example, Makinde [1] investigated the linear stability of hydromagnetic plane-Poiseuille flow at high Reynolds numbers by using the multideck asymptotic approach. Makinde and Mhone [2] reported the temporal development of small disturbances in magnetohydrodynamic Jeffery – Hamel flows through a convergent-divergent channel. Moreover, Makinde [3] examined the temporal development of small disturbances in a pressure-driven fluid flow through a channel filled with a saturated porous medium by using the Brinkman flow model. Adesanya [4] investigated the linear stability analysis of hydromagnetic fluid flow through a channel by using Adomian decomposition method. We refer interested readers to [5]-[6] for more on hydrodynamic stability analysis.

In all the above studies, the fluid viscosity has been assumed constant but viscosity is a very sensitive fluid property that varies with temperature, pressure or both in some cases. In view of this, Makinde [7] investigated the effect of variable viscosity on the hemodynamic stability analysis through a clear channel. However, the result may not be valid under certain disease conditions like formation of multiple stenosis during atherosclerosis. Therefore, as suggested in [8]-[9] the effect of stenosis can be captured in the model by taking the artery as a porous medium. Motivated by the results in [7]-[9], the specific objective of this paper is to investigate the effects of variations in viscosity and porous permeability on the flow stability which has not been accounted for in the previous models in literature.

2. Mathematical Formulation

Consider the flow of a variable viscous, incompressible fluid through a channel filled with a saturated porous materials. The dynamic viscosity of the fluid is assumed to vary with the channel width. If we employed a Cartesian coordinate system such that the x-axis corresponds to the flow direction and the y-axis is normal to it, then in 2-dimensions, the flow governing equations can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$$
$$=-\frac{\partial p}{\partial x}+\frac{2}{Re}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{Re}\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)-\frac{\mu\, u}{ReDa}, \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$
$$=-\frac{\partial p}{\partial y}+\frac{2}{Re}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{1}{Re}\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)-\frac{\mu\, v}{ReDa}. \quad (3)$$

where the following dimensionless variables and parameters have been used in (1)-(3)

$$\begin{array}{l} u=\frac{u'}{v_0}, \quad v=\frac{v'}{v_0}, \quad t=\frac{v_0 t'}{h}, \quad x=\frac{x'}{h}, \quad y=\frac{y'}{h}, \\ p=\frac{p'}{\rho v_0^2}, \quad \mu=e^{\beta\left(1-y^2\right)}, \quad Da=\frac{K}{h^2}, \quad G=Re\frac{dp}{dx}. \end{array} \quad (4)$$

In (4) v_0 is the characteristic fluid velocity, h is the half-width of the channel that represents the tube radius, K is the porous permeability, $\rho$is the fluid density, $\mu$is the dynamic viscosity, Da is the Darcy parameter, Re is the flow Reynolds number, (p, p') are the dimensionless and dimensional fluid pressure, (u, u') are the dimensionless and dimensional fluid velocity, $(x, x'), (y, y')$ are the dimensionless and dimensional Cartesian coordinates, β is the viscosity variation parameter.

The basic flow equation is given by

$$\frac{d^2u}{dy^2}=\frac{u}{Da}+2y\beta\frac{du}{dy}-Ge^{-\beta\left(1-y^2\right)}, \quad \frac{du}{dy}(0)=0, \quad u(1)=0 \quad (5)$$

to obtain the solution of the nonlinear equation (5), we assume $0<\beta\ll 2$, then by Taylor's expansion, we get

$$e^{-\beta\left(1-y^2\right)}=1+\beta\left(y^2-1\right)+\frac{\beta^2}{2}\left(y^2-1\right)^2+O\left(\beta^3\right), \quad (6)$$

with (6). It is more convenient to form a power series solution in the form

$$u(y)=\sum_{n=0}^{\infty}a_n y^n. \quad (7)$$

substituting (6)-(7) in (5), we get

$$0 = G - G\beta + G\beta y^2 + \frac{G\beta^2 y^4}{2} - G\beta^2 y^2 + \frac{G\beta^2}{2} + 2a_2 + 6a_3 y + 12a_4 y^2 - 2y\beta a_1 - 4a_2\beta y^2 - 6\beta a_3 y^3 - 8\beta a_4 y^4 - \frac{a_0}{Da} - \frac{a_1}{Da}y - \frac{a_2}{Da}y^2 - \frac{a_3}{Da}y^3 - \frac{a_4}{Da}y^4 + \cdots, \quad (8)$$

equating powers of y in (8) gives:

$$\begin{array}{ll} y^0: & G - G\beta + \frac{G\beta^2}{2} + 2a_2 - \frac{a_0}{Da} = 0, \\ y^1: & 6a_3 - 2\beta a_1 - \frac{a_1}{Da} = 0, \\ y^2: & G\beta - G\beta^2 + 12a_4 - 4\beta a_2 - \frac{a_2}{Da} = 0, \\ y^3: & -6a_3\beta - \frac{a_3}{Da} = 0, \\ y^4: & \frac{G\beta^2}{2} - 8a_4\beta - \frac{a_4}{Da} = 0. \end{array} \quad (9)$$

The system of equations (9) can be solved (with a_0 and a_1 fixed) to obtain:

$$\begin{aligned} a_2 =& \frac{a_0}{2Da} + \frac{G(\beta-1)}{2} - \frac{G\beta^2}{4}, \; a_3 = \left(\frac{1}{6Da} + \frac{\beta}{3}\right) a_1, \\ a_4 =& \frac{1}{12}\left(\frac{a_0}{2Da} + \frac{G(\beta-1)}{2} - \frac{G\beta^2}{4}\right)\left(\frac{1}{Da} + 4\beta\right) + \frac{G\beta(\beta-1)}{12}. \end{aligned} \quad (10)$$

Employing (10), we have

$$u = a_0 + a_1 y + \left(\frac{a_0}{2Da} + \frac{G(\beta-1)}{2} - \frac{G\beta^2}{4}\right) y^2 + \left(\frac{1}{6Da} + \frac{\beta}{3}\right) a_1 y^3 + \left(-\frac{G}{24Da} - \frac{G\beta}{4} + \frac{G\beta}{24Da} + \frac{G\beta^2}{4} - \frac{G\beta^2}{48Da} - \frac{G\beta^3}{12} + \frac{a_0}{24Da^2} + \frac{a_0\beta}{6Da}\right) y^4 + \cdots. \quad (11)$$

Using the boundary conditions we get

$$\begin{aligned} a_0 =& \frac{Da\left(13G\beta^2 - 24DaG(\beta-1) + 4DaG\beta^3 - 2G\beta - 12DaG\beta^2 + 2G + 12DaG\beta\right)}{48Da^2 + 24Da + 2 + 8Da}, \\ a_1 =& 0. \end{aligned} \quad (12)$$

Setting $\beta = 0.1$, $G = 2$ and $Da = 1$, the truncated series solution becomes

$$u(y) = 0.615661 - 0.512169y^2 - 0.103492y^4. \quad (13)$$

By using D'Alembert ratio test, the series solution (13) is evidently convergent.

Lemma 1. *If $u(x)$ satisfies the differential inequality*

$$(L+h)\,[u] \equiv u'' + g\,(x)\,u' + h\,(x)\,u \geq 0. \tag{14}$$

in an interval (a, b) with $h\,(x) \leq 0$, if g, h are bounded on every closed subinterval and if u assumes a nonnegative maximum value M at an interior point c, then $u\,(x) = M$.

Proof. see [10].

Theorem 1. *Suppose $u_1\,(y)\,,\ u_2\,(y)$ are solutions of (5) then $u_1\,(y) \equiv u_2\,(y)$.*

Proof. Let $u_1\,(y)\,,\ u_2\,(y)$ be any two independent solutions of (5), then we have

$$\frac{d^2u_1}{dy^2} - 2y\beta\frac{du_1}{dy} - \frac{u_1}{Da} = -Ge^{\beta\left(y^2-1\right)}, \tag{15}$$

and

$$\frac{d^2u_2}{dy^2} - 2y\beta\frac{du_2}{dy} - \frac{u_2}{Da} = -Ge^{\beta\left(y^2-1\right)}. \tag{16}$$

Suppose

$$w = u_1\,(y) - \ u_2\,(y)\,. \tag{17}$$

then (15) and (16) becomes

$$\frac{d^2w}{dy^2} - 2y\beta\frac{dw}{dy} - \frac{w}{Da}e^{\beta\left(1-y^2\right)} = 0. \tag{18}$$

By lemma 1, w cannot have its maximum in the interior hence $w = M = 0$.This implies that $u_1\,(y) = \ u_2\,(y)$ and as such the solution is unique.

3. Linear Stability Analysis

We superpose on the basic flow, small two-dimensional disturbances so that the velocity and pressure distribution are [4]

$$\begin{aligned} u\,(x,y,t) &= \bar{u}\,(y) + u'\,(x,y,t)\,, \\ v\,(x,y,t) &= v'\,(x,y,t)\,, \\ p\,(x,y,t) &= \bar{p}\,(x) + p'\,(x,y,t)\,, \end{aligned} \tag{19}$$

where $\bar{u}(y)$ is the solution to the basic mean equation $u'(x,y,t)$, $v'(x,y,t)$, $p'(x,y,t)$ are disturbances imposed on the flow. Substituting (19) in (1)-(3) and dropping all quadratic terms we obtain

$$\begin{aligned}
&\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0,\\
&\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+v'\frac{\partial \bar{u}}{\partial y}=-\frac{\partial p'}{\partial x}+\frac{2}{Re}\frac{\partial}{\partial x}\left(\mu\frac{\partial u'}{\partial x}\right)+\frac{1}{Re}\frac{\partial}{\partial y}\left(\mu\frac{\partial u'}{\partial y}\right)\\
&\qquad\qquad+\frac{1}{Re}\frac{\partial}{\partial y}\left(\mu\frac{\partial v'}{\partial x}\right)-\frac{\mu u'}{ReDa},\\
&\frac{\partial v'}{\partial t}+\bar{u}\frac{\partial v'}{\partial x}=-\frac{\partial p'}{\partial y}+\frac{2}{Re}\frac{\partial}{\partial y}\left(\mu\frac{\partial v'}{\partial y}\right)\\
&\qquad\qquad+\frac{1}{Re}\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\right)-\frac{\mu v'}{ReDa}.
\end{aligned} \tag{20}$$

To examine the nature of the stability, we assume a periodic solution of the Squire form.

$$\psi(x,y,t)=\varphi(y)\,e^{i\alpha(x-ct)}, \tag{21}$$

where $\varphi(y)$ is the amplitude function, α is the disturbances wave number which must be real for the solution to be bounded and

$$c=c_r+ic_i, \tag{22}$$

is the wave speed. Then (21) becomes

$$\psi(x,y,t)=\varphi(y)\,e^{i\alpha(x-c_r t)}e^{\alpha c_i t}. \tag{23}$$

with (23), the criterion condition for stability is that the wave is unstable if $\alpha c_i>0$, stable if $\alpha c_i<0$ and neutrally stable if$\alpha c_i=0$. Now the disturbance velocity component can be written as

$$\begin{aligned}
&u'=\frac{\partial\psi}{\partial y}=\varphi'(y)\,e^{i\alpha(x-ct)},\\
&v'=-\frac{\partial\psi}{\partial x}=-i\alpha\varphi(y)\,e^{i\alpha(x-ct)},\\
&p'=h(y)\,e^{i\alpha(x-ct)}.
\end{aligned} \tag{24}$$

Then substituting (24) in (20) and eliminating pressure, we obtain

$$\begin{aligned}
&(\bar{u}-c)\left(\varphi''(y)-\alpha^2\varphi(y)\right)-\varphi(y)\,\bar{u}''\\
&\quad=\frac{\mu}{i\alpha Re}\left(\varphi^{iv}(y)-\left(2\alpha^2+\frac{1}{Da}\right)\varphi''(y)+\left(\frac{\alpha^2}{Da}+\alpha^4\right)\varphi(y)\right)
\end{aligned}$$

$$+\frac{2\mu'}{i\alpha Re}\left(\varphi'''(y)-\frac{\varphi'(y)}{2Da}-\alpha^2\varphi'(y)\right)+\frac{\mu''}{i\alpha Re}\left(\varphi''(y)+\alpha^2\varphi(y)\right). \quad (25)$$

subject to the boundary conditions

$$\varphi(-1)=\varphi'(-1)=0, \quad (26)$$

$$\varphi(1)=\varphi'(1)=0. \quad (27)$$

Observed that as $Da\to\infty$, $\beta=0$ (25) reduces to the classical Orr-Summerfeld equation. If we rewrite (25), we get

$$\varphi^{iv}=4y\beta\varphi'''+\left(2\alpha^2-i\alpha Rece^{\beta\left(y^2-1\right)}+\frac{1}{Da}\right)\varphi''$$
$$+\left(i\alpha^3Rece^{\beta\left(y^2-1\right)}-\alpha^4+4y^2\alpha^2\beta^2-\frac{\alpha^2}{Da}\right)\varphi$$
$$+\left(i\alpha Re\bar{u}e^{\beta\left(y^2-1\right)}-4y^2\beta^2\right)\varphi''-i\alpha Ree^{\beta\left(y^2-1\right)}\left(\alpha^2\bar{u}+\bar{u}''\right)\varphi$$
$$-2y\beta\left(\frac{1}{Da}+2\alpha^2\right)\varphi'. \quad (28)$$

Converting the differential equation (28) to its equivalent integral equation, we then seek a series solution of the form

$$\varphi(y)=\sum_{n=0}^{\infty}\varphi_n(y). \quad (29)$$

Using (29) after a few manipulations, we get the zeroth component as

$$\varphi_0(y)=\int_{-1}^{y}\int_{-1}^{y}\varphi''(-1)dydy, \quad (30)$$

while the other component of the series can be obtained uniquely by using the recurrence relations

$$\varphi_{n+1}(y)=\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\varphi'''(-1)dydydy$$
$$+\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\left[4y\beta\varphi_n'''+\left(2\alpha^2-i\alpha Rece^{\beta\left(y^2-1\right)}+\frac{1}{Da}\right)\varphi_n''\right]dydydydy$$

$$+\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\left[\left(i\alpha Re\bar{u}e^{\beta\left(y^2-1\right)}-4y^2\beta^2\right)\varphi_n''-i\alpha Ree^{\beta\left(y^2-1\right)}\left(\alpha^2\bar{u}+\bar{u}''\right)\varphi_n\right.$$
$$\left.-2y\beta\left(\frac{1}{Da}+2\alpha^2\right)\varphi_n'\right]dydydydy$$
$$+\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\int_{-1}^{y}\left[\left(i\alpha^3 Rece^{\beta\left(y^2-1\right)}-\alpha^4+4y^2\alpha^2\beta^2-\frac{\alpha^2}{Da}\right)\varphi_n\right]dydydydy. \quad (31)$$

Then the truncated series solution

$$\varphi\left(y\right)=\sum_{n=0}^{k}\varphi_n\left(y\right) \quad (32)$$

gives the approximate solution, where k is the truncation point. Following [4] to find the eigenvalues, we solved the partial sum (32) utilizing the boundary conditions (27). This resulted in two equations as functions of b_0 and b_1. By solving both equations for one of the unknown b_1 and equating them to each other, we can eliminate both b_0 and b_1 yielding an equation that can be solved for c, the wave speed. Due to large size of mathematica output, we shall only present the numerical results.

4. Discussion of Results

The Table 1 shows the stability analysis for variations in the Darcy number. From the result, the imaginary part of the wave speed is observed to increase with increasing value of the Darcy parameter, Da, this implies instability on the flow and hence a decrease in the Darcy parameter has stabilizing effect on the flow. This behaviour is in perfect agreement with previously obtained results in [3]. From Table 2, it is observed that the imaginary part of the wave speed is reduced with increase in the viscosity variation parameters; this shows that increase in viscosity stabilizes the flow.

The growth of plaque represents a decrease in the radius of the tube and decrease in Darcy parameter. From Figure 1, it is observed that decrease in the Darcy parameter causes a decrease in the velocity maximum and if this continues then a case of zero flow could happen. Also in Figure 2, it is observed that an increase in the hematocrit parameter leads a decrease in the maximum velocity. This is due to the fact that as the viscosity increases the fluid becomes hyper-viscous

Da	c_r	c_i
0.1	0.169662	-0.000825205i
0.2	0.296112	-0.000285822i
0.3	0.389816	-0.000106027i
0.4	0.461463	-0.0000161301i
0.5	0.517876	0.0000378082i

Table 1: Computation showing variations in wave speed $\beta = 0.2$, $\alpha = 1$, $Re = 10^4$

β	c_r	c_i
0.01	0.184567	-0.000772511i
0.1	0.169662	-0.000825205i
0.25	0.147963	-0.000911483i
0.5	0.119958	-0.00103636i
0.75	0.101419	-0.00110392i

Table 2: Computation showing variations in wave speed $Da = \alpha = 1$, $Re = 10^4$

5. Conclusion

We have studied the temporal stability of small disturbances imposed on blood flow due to stenosis by using Adomian decomposition method. The result shows that decrease in Darcy parameter decreases the flow velocity and has stabilizing effect on the flow, while increase in the hematocrit parameter reduces the fluid velocity and stabilizes the flow.

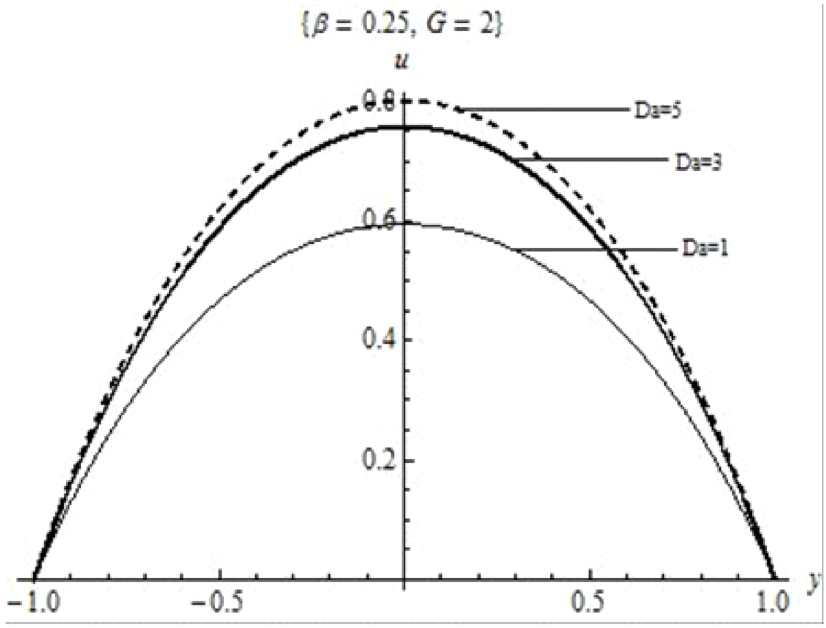

Figure 1: Velocity profile for variation in the Darcy parameter

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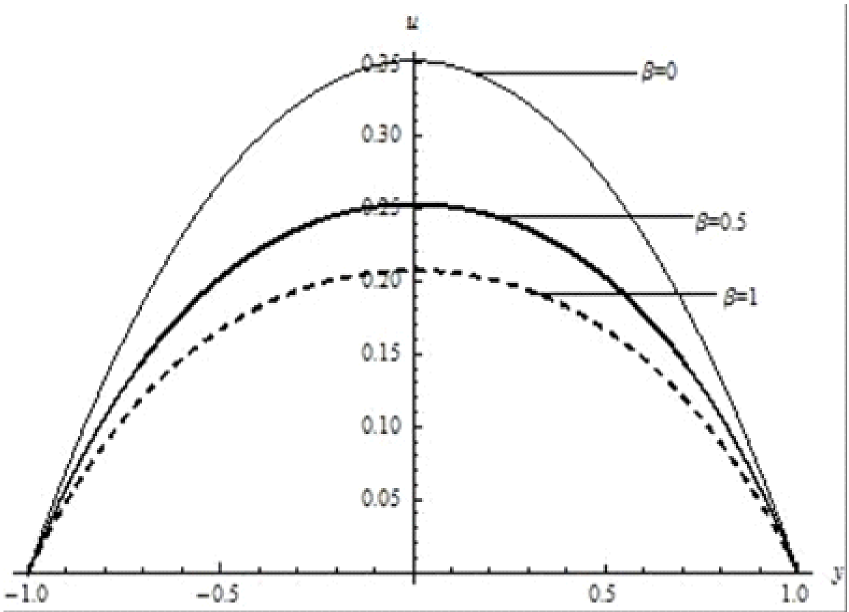

Figure 2: Velocity profile for variation in the hematocrit parameter

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