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ULTRASUBHARMONIC PERIODIC ORBITS IN SEMICONDUCTOR LASER EQUATIONS

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Abstract: We use a third-order phase equation associated with the Maxwell-Bloch equations to investigate the behavior of a single-mode semiconductor laser subject to injection and frequency detuning. The weakly nonlinear phase equation is analyzed using standard regular perturbation techniques. We derive an approximate Poincaré map to study the organization of the ultrasubharmonic periodic orbits when the frequency of the injected beam is a rational multiple of the natural frequency of the free-running laser.

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1. Introduction

Semiconductor laser diodes are nonlinear, or nonohmic devices, whose current-voltage characteristic is not the straight line prescribed by Ohm's law. The operation of a single-mode semiconductor laser is governed by the rate equations

$$\begin{aligned} Z' &= ZN(1-ib), \\ TN' &= P - N - P|Z|^2(1+2N), \end{aligned} \tag{1}$$

where Z is the complex electric field inside the laser cavity and N, the number

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of carriers (electrons and holes) above threshold.

The parameters T and b represent quantities associated with the laser's material characteristics, and P corresponds to the intensity $(\frac{J}{J_{th}} - 1)$ of the pumping current above threshold.

The equations in (1) make up a nonlinear system whose behavior is well explained by the theory of dynamical systems. However, when external light $\eta e^{-i\Omega t}$ of amplitude η and relative frequency Ω is injected into the laser cavity, the situation changes drastically. The modified equations that model this more interesting set-up are obtained by adding the coupling term

$$\begin{pmatrix} \eta e^{-i\Omega t} \\ 0 \end{pmatrix},$$

to the complex vector field. The resulting system

$$\begin{aligned} Z' &= ZN(1 - ib) + \eta e^{-i\Omega t}, \\ TN' &= P - N - P|Z|^2(1 + 2N) \end{aligned} \tag{2}$$

can be averaged and the averaged equations have been studied [7], [12], [14], [6], [8] using the tools of bifurcation theory.

Kovacic et al. [5] proved the existence of homoclinic orbits for this system of differential equations. Evidence of anomalous pushing of the laser's frequency away from the external reference frequency was presented by Politi et al. [10]. Numerical investigations by several authors [11], [6], [2], [3], [9], [1], [13] have shed light on such phenomena as the coexistence of conservative and dissipative behavior in the same phase space, periodic and quasi-periodic motions on a torus born of a Hopf bifurcation, period-doubling cascading, to name a few.

In this work, our analysis will center on a weakly nonlinear phase equation that we will extract from (2). Our focus will be on how resonance between the forcing frequency Ω and the laser's internal frequency ω influences the organization of the periodic orbits in the three-dimensional phase space.

This paper is organized as follows: In Section 2 we use the amplitude E and phase Ψ of the electric field to rewrite the rate equations in real form. Under appropriate assumptions on the parameters of the system, we derive a third-order differential equation for the phase of the electric field, the so-called phase equation. In section 3 we derive an approximate Poincaré map to study the periodic solutions of the phase equation for small rational values of the ratio $\frac{\Omega}{\omega}$. We summarize our results in the last section of the paper.

2. Phase Equation

Using the amplitude E and phase ψ of the complex electric field, we use the change of variables $Z = Ee^{i\psi}$ to cast the complex rate equations (2) in the real form

$$\begin{aligned} E' &= NE + \eta\cos(\psi + \Omega t) \\ TN' &= P - N - P(1+2N)E^2 \\ \psi' &= -bN - \eta\frac{\sin(\psi+\Omega t)}{E}. \end{aligned} \tag{3}$$

We observe that in the absence of injection, the parameter η is equal to 0, meaning that the unperturbed laser is governed by the subsystem

$$\begin{aligned} E' &= NE \\ TN' &= P - N - P(1+2N)E^2, \end{aligned}$$

which admits a circle of fixed points in the complex $Z-$plane given by $(E,N) = (1,0)$. Linearization about $(E,N) = (1,0)$ reveals that the unit circle is attracting with nearby motion characterized by damped oscillation with frequency $\omega = \sqrt{\frac{2P}{T}}$ in a direction perpendicular to the $Z-$plane. What becomes of this degenerate circle of fixed points when we restore the injection?

Kovanis, Erneux et al. [2], [3] use the fact that the linewidth enhancement factor b is large to come up with an equation for ψ that governs the dynamics in a small neighborhood of the unit circle.

In [3] the scaling

$$s = t\sqrt{2P/T}, \quad E = 1 + a/b, \quad N = n\sqrt{2P/T}\frac{1}{b}$$

is used to turn (3) into

$$\begin{aligned} \dot{a} &= n + \Lambda\cos(\psi + \Delta s) + O(\frac{1}{b}) \\ \dot{n} &= -a - \zeta n + O(\frac{1}{b}) \\ \dot{\psi} &= -n + O(\frac{1}{b}). \end{aligned} \tag{4}$$

The dot stands for differentiation with respect to the re-scaled time parameter s. The new parameters are related to η, P, b, T and Ω as follows:

$$\omega \equiv \sqrt{2P/T},$$

$$\Lambda = \eta b/\omega,$$
$$\Delta \equiv \Omega/\omega,$$
$$\zeta \equiv \omega \frac{1+2P}{2P}.$$

We have control over the parameter Λ, since the injection strength η can be made as small as desired. The value of the pumping current P lies in the interval $[0, 1)$. The magnitude for T is around 10^3 and b usually varies between 3 and 14. As a result, the new parameters ω and ζ are small and we can adjust the new detuning parameter Δ by choosing Ω appropriately.

An inspection of Equation (4) suggests that, for Λ nonzero but small, the new variables a and n oscillate with frequency approximately equal to 1. This means that the E and N components of (3) continue to oscillate with frequency near ω, and a qualitative description of the flow only requires an understanding of how the electric field vector rotates in the complex $Z-$plane. We will henceforth focus our attention on the behavior of the phase ψ.

After dropping the small $O(\frac{1}{b})$ terms, differentiation of the ψ component of (4) yields the third-order differential equation

$$\dddot{\psi} + \zeta\ddot{\psi} + \dot{\psi} = \Lambda\cos(\psi + \Delta s). \tag{5}$$

Notice that $\Delta = \frac{m}{n}$, m and n relatively prime integers, corresponds to the resonance relation $n\Omega = m\omega$ between the natural frequency of the free-running laser and the frequency of the beam injected into the laser cavity.

Our investigation will focus on the detuning parameter regime where Δ is less than Λ, with both parameters small. We will refer to this as the low detuning regime. We will restrict Δ to rational values $\frac{m}{n} < \Lambda$ in order to investigate the organization of the ultrasubharmonics of order m, n.

3. The Low Detuning Regime

Our study of resonance in the low detuning regime will focus on the ultrasubharmonics of the system associated with

$$\psi''' + \zeta\psi'' + \psi' = \Lambda\cos(\psi + \Delta t),$$

which is Equation (5) written with t instead of s for notational convenience, and where prime stands for differentiation with respect to t.

We rewrite this equation as the 3-dimensional system

$$u' = v$$

$$\begin{aligned} v' &= -u - \zeta v + \Lambda \cos(\psi + \Delta t) \\ \psi' &= u. \end{aligned}$$

The change of variables

$$u = \sigma, v = \rho, \psi = \tau - \rho,$$

casts this system in the form

$$\begin{aligned} \sigma' &= \rho \\ \rho' &= -\sigma - \zeta\rho + \Lambda \cos(\tau - \rho + \Delta t) \\ \tau' &= -\zeta\rho + \Lambda \cos(\tau - \rho + \Delta t). \end{aligned}$$

Using $\zeta = \delta\Lambda$ and polar coordinates

$$\sigma = r\cos\theta, \quad \rho = r\sin\theta,$$

we get

$$\begin{aligned} r' &= \Lambda[\sin\theta \cos(\tau - r\sin\theta + \Delta t) - \delta r \sin^2\theta] \\ \tau' &= \Lambda[-\delta r \sin\theta + \cos(\tau - r\sin\theta + \Delta t)] \\ \theta' &= -1 - \Lambda\delta \sin\theta\cos\theta + \frac{\Lambda}{r}\cos\theta\cos(\tau - r\sin\theta + \Delta t). \end{aligned} \tag{6}$$

Equations (6) form a two-frequency system. To analyze it, we derive an approximate Poincaré map which allows us to see precisely how the ultrasubharmonic periodic orbits depend on Δ.

The construction of the approximate Poincaré map follows the method outlined in [15]. First, we introduce a new variable Φ to write (6) as an autonomous system:

$$\begin{aligned} r' &= \Lambda[\sin\theta \cos(\tau - r\sin\theta + \Phi) - \delta r \sin^2\theta] \\ \tau' &= \Lambda[-\delta r \sin\theta + \cos(\tau - r\sin\theta + \Phi)] \\ \theta' &= -1 - \Lambda\delta \sin\theta\cos\theta + \frac{\Lambda}{r}\cos\theta\cos(\tau - r\sin\theta + \Phi) \\ \Phi' &= \Delta. \end{aligned} \tag{7}$$

The system in (7) is of the form

$$x' = f(x) + \Lambda g(x),$$

with

$$x = \begin{pmatrix} r \\ \tau \\ \theta \\ \Phi \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ \Delta \end{pmatrix},$$

and

$$g(x) = \begin{pmatrix} \sin\theta\cos(\tau - r\sin\theta + \Phi) - \delta r \sin^2\theta \\ -\delta r\sin\theta + \cos(\tau - r\sin\theta + \Phi) \\ -\delta\sin\theta\cos\theta + \frac{1}{r}\cos\theta\cos(\tau - r\sin\theta + \Phi) \\ 0 \end{pmatrix}.$$

For Λ small and t finite, approximate solutions of (7) are of the form

$$x(t) = x_0(t) + \Lambda x_1(t) + O(\Lambda^2),$$

where $x_0(t)$ is known and is given by

$$x_0(t) = \begin{pmatrix} r_0 \\ \tau_0 \\ -t + \theta_0 \\ \Delta t + \Phi_0 \end{pmatrix}.$$

To find $x_1(t)$, we differentiate the solution $x(t)$ to obtain

$$\begin{aligned} x'(t) &= x_0'(t) + \Lambda x_1'(t) + O(\Lambda^2) \\ &= f(x_0(t) + \Lambda x_1(t) + O(\Lambda^2)) + \Lambda g(x_0(t) + \Lambda x_1(t) + O(\Lambda^2)). \end{aligned}$$

Expanding $g(x(t))$ around $x_0(t)$ and using the fact that f is constant yields

$$x_0'(t) + \Lambda x_1'(t) + O(\Lambda^2) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ \Delta \end{pmatrix} + \Lambda g(x_0(t)) + O(\Lambda^2),$$

and, therefore,

$$x_1'(t) = g(x_0(t)),$$

or

$$x_1(t) = \int_0^t g(x_0(s))ds.$$

So, over a finite time, solutions of (7) are given by

$$x(t) = x_0(t) + \Lambda \int_0^t g(x_0(s))ds + O(\Lambda^2),$$

or

$$\begin{pmatrix} r(t) \\ \tau(t) \\ \theta(t) \\ \Phi(t) \end{pmatrix} = \begin{pmatrix} r_0 \\ \tau_0 \\ -t+\theta_0 \\ \Delta t+\Phi_0 \end{pmatrix} +$$

$$\Lambda \begin{pmatrix} \int_0^t [\sin(-s+\theta_0)\cos(\tau_0 - r_0\sin(-s+\theta_0)+\Delta s+\Phi_0) - \delta r_0 \sin^2(-s+\theta_0)]ds \\ \int_0^t [-\delta r_0 \sin(-s+\theta_0) + \cos(\tau_0 - r_0\sin(-s+\theta_0)+\Delta s+\Phi_0)]ds \\ \int_0^t [-\delta \sin(-s+\theta_0)\cos(-s+\theta_0) + \\ \frac{1}{r_0}\cos(-s+\theta_0)\cos(\tau_0 - r_0\sin(-s+\theta_0)+\Delta s+\Phi_0)]ds \\ 0 \end{pmatrix} + O(\Lambda^2) \quad (8)$$

Since $\Phi(t) = \Delta t + \Phi_0$ in (8) , we can define a global cross section Σ to the vector field in (7) as follows:

$$\Sigma^{\Phi_0} = \left\{ \begin{pmatrix} r \\ \tau \\ \theta \\ \Phi \end{pmatrix} : \Phi = \Phi_0 \right\}.$$

Solutions leaving the cross section based at $\Phi = \Phi_0$ at time $t = 0$ come back to it at a time $t = \frac{2\pi}{\Delta}$. Thus our approximate Poincaré map is given by

$$P_\Lambda : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r(0) \\ \tau(0) \\ \theta(0) \end{pmatrix} \longmapsto \begin{pmatrix} r(\frac{2\pi}{\Delta}) \\ \tau(\frac{2\pi}{\Delta}) \\ \theta(\frac{2\pi}{\Delta}) \end{pmatrix},$$

with m^{th} iterate given by

$$P_\Lambda^m : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r(0) \\ \tau(0) \\ \theta(0) \end{pmatrix} \longmapsto \begin{pmatrix} r(\frac{2m\pi}{\Delta}) \\ \tau(\frac{2m\pi}{\Delta}) \\ \theta(\frac{2m\pi}{\Delta}) \end{pmatrix}.$$

Finally we obtain

$$P_\Lambda^m : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 - \frac{2m\pi}{\Delta} \end{pmatrix}$$

$$+\Lambda \begin{pmatrix} \int_0^{\frac{2m\pi}{\Delta}} [\sin(-t+\theta_0)\cos(\tau_0 - r_0\sin(-t+\theta_0)+\Delta t+\Phi_0) \\ \qquad - \delta r_0 \sin^2(-t+\theta_0)]dt \\ \int_0^{\frac{2m\pi}{\Delta}} [-\delta r_0 \sin(-t+\theta_0) + \cos(\tau_0 - r_0\sin(-t+\theta_0)+\Delta t+\Phi_0)]dt \\ \int_0^{\frac{2m\pi}{\Delta}} [-\delta \sin(-t+\theta_0)\cos(-t+\theta_0) \\ \qquad + \frac{1}{r_0}\cos(-t+\theta_0)\cos(\tau_0 - r_0\sin(-t+\theta_0)+\Delta t+\Phi_0)]dt \end{pmatrix}$$

$$+ O(\Lambda^2),$$

where $r(0) = r_0, \tau(0) = \tau_0, \theta(0) = \theta_0$.

Assuming that Δ satisfies the relation

$$\Delta n = m, \text{for integers } m \text{ and } n,$$

we can compute the above integrals to get the map

$$P_\Lambda^m : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 - \Lambda\delta n\pi r_0 - \Lambda 2n\pi \sin(\tau_0 + \frac{m}{n}\theta_0 + \Phi_0) j'_{-\frac{m}{n}}(r_0) \\ \tau_0 + \Lambda 2n\pi \cos(\tau_0 + \frac{m}{n}\theta_0 + \Phi_0) j_{-\frac{m}{n}}(r_0) \\ \theta_0 - 2n\pi + \Lambda \frac{2n\pi}{r_0^2} \cos(\tau_0 + \frac{m}{n}\theta_0 \\ \qquad + \Phi_0)[\frac{m}{n} j_{-\frac{m}{n}}(r_0) + \frac{\sin(\frac{m\pi}{n})}{\pi}] \end{pmatrix}$$

$$+ O(\Lambda^2),$$

where $j_w(r)$ is the Anger function of order w and is equal to the Bessel function $J_w(r)$ when w is an integer [4], and $j'_w(r)$ is the derivative of $j_w(r)$ with respect to r.

The Poincaré map P_Λ^m illustrates how strongly the existence of periodic orbits depends on the ratio Δ of the two frequencies. Let us set $\Delta = \frac{k}{l}$, where $l > k$, and k and l are relatively prime integers. The original Poincaré map is of the form

$$P_\Lambda : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 + O(\Lambda) \\ \tau_0 + O(\Lambda) \\ \theta_0 - \frac{2l\pi}{k} + O(\Lambda) \end{pmatrix}.$$

As the second iterate, we have

$$P_\Lambda^2 : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 + O(\Lambda) \\ \tau_0 + O(\Lambda) \\ (\theta_0 - \dfrac{2l\pi}{k}) - \dfrac{2l\pi)}{k} + O(\Lambda) \end{pmatrix}.$$

It is not until the k^{th} iterate that we obtain

$$P_\Lambda^k : \Sigma^{\Phi_0} \to \Sigma^{\Phi_0},$$

$$\begin{pmatrix} r_0 \\ \tau_0 \\ \theta_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 + O(\Lambda) \\ \tau_0 + O(\Lambda) \\ \theta_0 - k\dfrac{2l\pi}{k} + O(\Lambda) \end{pmatrix} = \begin{pmatrix} r_0 + O(\Lambda) \\ \tau_0 + O(\Lambda) \\ \theta_0 + O(\Lambda) \end{pmatrix}.$$

So for $m < k$, P_Λ^m cannot have any fixed points if Λ is sufficiently small. It is likewise, for m between k and $2k$.

Given that (6) is a two-frequency system, what we have is the following: if the system is driven at a set frequency Δ, the m^{th} iterate of the Poincaré map P_Λ^m possesses fixed points only if $m = \Delta n$, m and n being integers. This is a resonance condition relating the forcing frequency Δ to the frequency -1 of the unperturbed system. When translated to the setting of the injected laser system, this connects the relative frequency Ω of the injected beam to the natural frequency ω of the semiconductor laser through the resonance relation $n\Omega = m\omega$.

The map P_Λ^m can be seen to be two-dimensional if we introduce the variable $y_0 = \tau_0 + \frac{m}{n}\theta_0$ and set the cross-section $\Phi_0 = 0$. To order Λ, we obtain the new map

$$\begin{pmatrix} r_0 \\ y_0 \end{pmatrix} \longmapsto \begin{pmatrix} r_0 - \Lambda\delta n\pi r_0 - \Lambda 2n\pi \sin(y_0) j'_{-\frac{m}{n}}(r_0) \\ y_0 + \Lambda 2\pi \cos(y_0)\left[n j_{-\frac{m}{n}}(r_0) + \dfrac{m^2}{n r_0^2} j_{-\frac{m}{n}}(r_0) + \dfrac{m}{r_0^2} \dfrac{\sin(\frac{m\pi}{n})}{\pi} \right] \end{pmatrix}.$$

When $\delta = 0$, the zeroes of the function $j'_{-\frac{m}{n}}$ form one-dimensional invariant manifolds that run perpendicular to the invariant manifolds given by the lines $y_0 = (2n+1)\frac{\pi}{2}$ in the $r_0 - y_0$ plane. Thus the phase portrait for the map follows a simple rectangular grid pattern where horizontal and vertical invariant lines intersect to form heteroclinic 4-cycles. Inside each 4-cycle is a fixed point of center type, i.e., a fixed point surrounded by a continuum of periodic orbits. This center fixed point becomes an attracting focus once δ is set to a small positive value. These attracting fixed points of the map correspond to the attracting periodic orbits of the third-order phase equation we extracted from the original rate equations.

4. Conclusion

Our approach to the rate equations has been through a third-order phase equation. We were able to derive an approximate Poincaré map to study the ultra-subharmonics of this phase equation in the low detuning regime. We determined that, for small rational values $\frac{m}{n}$ of the ratio $\Delta = \frac{\Omega}{\omega}$, we have periodic solutions of period $2n\pi$. These periodic solutions are fixed points of the m^{th} iterate of the Poincaré map, which means that, viewed in a three-dimensional phase space, points on these orbits cross a Poincaré section m times before returning to their initial positions. The Poincaré map, to order Λ, admits a network of interlocking horizontal and vertical invariant manifolds. These invariant manifolds form heteroclinic 4-cycles which, upon restoration of higher-order terms, are expected to break down in a way that gives rise in the full map to transverse homoclinic trajectories, a signature of chaotic dynamics.

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