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STABILITY ANALYSIS AND SIMULATION OF AN AGE-STRUCTURED HEPATITIS B MODEL WITHOUT VERTICAL TRANSMISSION

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Abstract: This paper deals with the analysis of the local and global stability of an modified age-structured model for the transmission dynamics of hepatitis B without vertical transmission.

The proposed model takes into account the additional mortality rate associated to the desease and incorporates vaccination strategy.

AMS Subject Classification: 92B05, 92C60, 92D25, 92D30, 34D25
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1. Introduction

This paper deals with the analysis of the local and global stability of an modified age-structured model for the transmission dynamics of hepatitis B without

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vertical transmission. Hepatitis B virus (HBV) is widespreads in many parts of the world, especially in Africa (Centers for disease control and prevention (CDC)[1]. By the estimation of World Health Organisation (WHO)[2], about Two billion people have been infected with HBV during their life, about 30 per cent of the world population. An estimate of 600,000 persons died each year due to the acute or chronic consequences of the virus [2]. The HBV is 50 to 100 times more contaminant than HIV. Hepatitis B is endemic in Africa, 18 percent of chronic carriers live in Africa, about 65 million in 2007. In this model we study the dynamic of hepatitis B without vertical transmition, but we consider the additional mortality rate associated to the desease. Age structure a added to the continuous time t brings some improvements in the comprehension of the disease dynamics. We follow methods of L. Zou, S. Ruan and W. Zhang.[3], C. Castillo-Chavez and Z. Feng.[4], X.Z. Li and J.X Liu [5] or X.Z. Li and B. Fang [6], Inaba [7] for quantitative (wellposedness with semigroup theory). One can see also C. Castillo-Castillo and Z. Feng[4], Greenhalg [8] for a good review.

We study impact on basic reproductive number R_0 and net reproductive number $R(\Psi)$ in the dynamic of disease. Our goal is to show the importance of considering the additional mortality rate associated to the desease for quanlitative and quantitative results in the dynamics of disease. We study the local and gobal stability of Disease Free Equilibrium (DFE), the local stability of Endemic Equilibrium (EE) and show some simulations.

The manuscript is organised as follow: second section is devoted to Problem formulations, third section to Abstract formulation, fourth section to DFE and local stability, fifth section to Existence of EE and local stability, sixth section to Global stability of DFE, and last section to Simulation and conclusion.

2. Problem Formulations

We consider a general age structured model describing the dynamics of transmission of hepatitis B without vertical transmission. A population of size N(a,t) is stratified into six compartments, namely: susceptible S(a,t), latently infected E(a,t), acutely infectious I(a,t), carrier C(a,t), recovered R(a,t) and vaccinated V(a,t) with age distribution at time t.

The variables and model structured are describe in Figure 1.

The parameter used are described as follows:

- $b(a)$ birth rate.
- $\mu(a)$ natural mortality rate.

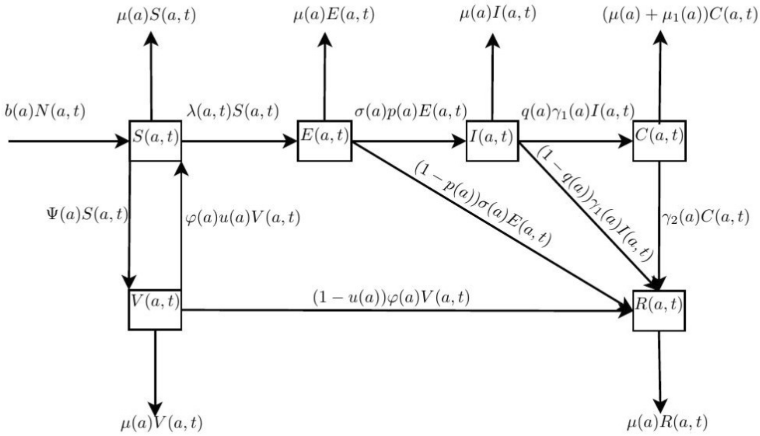

Figure 1: Model of HBV without vertical transmission

- $\mu_1(a)$ HBV-related mortality rate .
- ϵ reduced transmission rate from chronic carriers compared to acute infections.
- $\sigma(a)$ rate moving from latent to acute.
- $\gamma_1(a)$ rate moving from acute to carrier.
- $\gamma_2(a)$ rate moving from carrier to recovered.
- $\Psi(a)$ vaccination rate against hepatitis B.
- $q(a)$ probability an individual fails to clear an acute infection and develops to carrier state.
- $\varphi(a)$ rate moving from vaccinated to susceptible.
- $p(a)$ probability an individual fails to clear an latent infection and develops to acute infection state.
- $u(a)$ probability an individual fails to clear vaccinated and develops to susceptible state.

Under the above assumptions and parameters, the dynamic of the desease can be described by the following system of partial differential equations:

$$\begin{cases} \frac{\partial S(a,t)}{\partial \mathrm{t}} + \frac{\partial S(a,t)}{\partial \mathrm{a}} = b(a)N(a,t) + \varphi(a)u(a)V(a,t) \\ \qquad\qquad - (\mu(a) + \lambda(a,t) + \Psi(a))S(a,t) \\ \frac{\partial E(a,t)}{\partial \mathrm{t}} + \frac{\partial E(a,t)}{\partial \mathrm{a}} = \lambda(a,t)S(a,t) - (\mu(a) + \sigma(a))E(a,t) \\ \frac{\partial I(a,t)}{\partial \mathrm{t}} + \frac{\partial I(a,t)}{\partial \mathrm{a}} = \sigma(a)p(a)E(a,t) - (\mu(a) + \gamma_1(a))I(a,t) \\ \frac{\partial C(a,t)}{\partial \mathrm{t}} + \frac{\partial C(a,t)}{\partial \mathrm{a}} = q(a)\gamma_1(a)I(a,t) - (\mu(a) + \mu_1(a) + \gamma_2(a))C(a,t) \\ \frac{\partial R(a,t)}{\partial \mathrm{t}} + \frac{\partial R(a,t)}{\partial \mathrm{a}} = -(1 - u(a))\varphi(a)V(a,t) + (1 - p(a))\sigma(a)E(a,t) \\ \qquad\qquad + (1 - q(a))\gamma_1(a)I(a,t) + \gamma_2(a)C(a,t) - \mu(a)R(a,t) \\ \frac{\partial V(a,t)}{\partial \mathrm{t}} + \frac{\partial V(a,t)}{\partial \mathrm{a}} = \Psi(a)S(a,t) - (\mu(a) + \varphi(a))V(a,t) \end{cases} \tag{1}$$

with initial and boundary conditions:

$$\begin{cases} S(0,t) = \int_{a_1}^{a_2} b(a)N(a,t)\mathrm{da}; \\ E(0,t) = I(0,t) = C(0,t) = R(0,t) = V(0,t) = 0; \\ S(a,0) = S_0(a); E(a,0) = E_0(a); I(a,0) = I_0(a); C(a,0) = C_0(a); \\ R(a,0) = R_0(a); V(a,0) = V_0(a). \end{cases}$$

Where a_1 and a_2 are respectively the minimum and maximum age of procreation.

The total population is:

$$N(a,t) = S(a,t) + E(a,t) + I(a,t) + C(a,t) + R(a,t) + V(a,t).$$

By summing all equations of system (1), we obtain the following equations for the total population $N(a,t)$:

$$\frac{\partial N(a,t)}{\partial \mathrm{t}} + \frac{\partial N(a,t)}{\partial \mathrm{a}} = b(a)N(a,t) - \mu(a)N(a,t) - \mu_1(a)C(a,t).$$

$$N(0,t) = \int_{a_1}^{a_2} b(a)N(a,t)\mathrm{da}.$$

Those relations show that the population dynamics is affected by the desease.

Following [3] one can express the force of infection by:

$$\lambda(a,t) = k(a)\int_0^{a_+} \tilde{\beta}(a')(\frac{I(a',t)+\epsilon C(a',t)}{N(a',t)})\mathrm{d}a'$$

Where a_+ is the maximum age of individual, the function $k(a)$ and $\tilde{\beta}(a')$ are respectively the age-specific (average) probability of becoming infected through contact with infectious individuals and the age-specific per-capita contact/activity rate.

Let:

$$s(a,t) = \frac{S(a,t)}{N(a,t)}, \quad e(a,t) = \frac{E(a,t)}{N(a,t)}, \quad i(a,t) = \frac{I(a,t)}{N(a,t)},$$

$$c(a,t) = \frac{C(a,t)}{N(a,t)}, \quad r(a,t) = \frac{R(a,t)}{N(a,t)}, \quad v(a,t) = \frac{V(a,t)}{N(a,t)}.$$

Then we obtain the following normalised system of partial differential equations:

$$\begin{cases}
\frac{\partial s(a,t)}{\partial \mathrm{t}} + \frac{\partial s(a,t)}{\partial \mathrm{a}} = b(a) + \varphi(a)u(a)v(a,t) \\
\qquad - (-\mu_1(a)c(a,t) + k(a)\lambda(t) + \Psi(a) + b(a))s(a,t) \\
\frac{\partial e(a,t)}{\partial \mathrm{t}} + \frac{\partial e(a,t)}{\partial \mathrm{a}} = k(a)\lambda(t)s(a,t) - (b(a) - \mu_1(a)c(a,t) + \sigma(a))e(a,t) \\
\frac{\partial i(a,t)}{\partial \mathrm{t}} + \frac{\partial i(a,t)}{\partial \mathrm{a}} = \sigma(a)p(a)e(a,t) - (b(a) - \mu_1(a)c(a,t) + \gamma_1(a))i(a,t) \\
\frac{\partial c(a,t)}{\partial \mathrm{t}} + \frac{\partial c(a,t)}{\partial \mathrm{a}} = q(a)\gamma_1(a)i(a,t) - (b(a) - \mu_1(a)c(a,t) + \mu_1(a) \\
\qquad + \gamma_2(a))c(a,t) \\
\frac{\partial r(a,t)}{\partial \mathrm{t}} + \frac{\partial r(a,t)}{\partial \mathrm{a}} = -(1-u(a))\varphi(a)v(a,t) \\
\qquad + (1-p(a))\sigma(a)e(a,t) + (1-q(a))\gamma_1(a)i(a,t) \\
\qquad + \gamma_2(a)c(a,t) - (b(a) - \mu_1(a)c(a,t))r(a,t) \\
\frac{\partial v(a,t)}{\partial \mathrm{t}} + \frac{\partial v(a,t)}{\partial \mathrm{a}} = \Psi(a)s(a,t) - (b(a) - \mu_1(a)c(a,t) + \varphi(a))v(a,t) \\
\lambda(t) = \int_0^{a_+} \tilde{\beta}(a')(i(a',t) + \epsilon c(a',t))\mathrm{d}a'
\end{cases} \qquad (2)$$

with initial and boundary conditions:

$$s(0,t) = 1; e(0,t) = i(0,t) = c(0,t) = r(0,t) = v(0,t) = 0$$

3. Abstract Formulation

Let X be a Banach space defined by $X := (L^1(0,a_+))^6$, endowed with the norm $\| \phi \| = \sum_{j=1}^{6} \| \phi_j \|$ for $\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a), \phi_6(a))^T \in X$, where $\| . \|$ is the norm of $L^1(0,a_+)$. The state space of system (2) is:

$$\Omega := \{(s,e,i,c,r,v) \in X_+ / 0 \leqslant s+e+i+c+r+v \leqslant 1\},$$

where $X_+ = (L^1_+(0,a_+))^6$, and $L^1_+(0,a_+)$ denotes the positive cone of $L_1(0,a_+)$.

Let A be a linear operator defined by

$$(A\phi)(a) = (A_1, A_2, A_3, A_4, A_5, A_6)^T,$$

where:

$$A_1 = \left(\tfrac{-d\phi_1}{da} - (b(a)+\Psi(a))\phi_1, 0,0,0,0, \varphi(a)u(a)\phi_6\right),$$

$$A_2 = \left(0, \tfrac{-d\phi_2}{da} - (b(a)+\sigma(a))\phi_2, 0,0,0,0\right),$$

$$A_3 = \left(0, \sigma(a)p(a)\phi_2, \tfrac{-d\phi_3}{da} - (b(a)+\gamma_1(a))\phi_3, 0,0,0\right),$$

$$A_4 = \left(0,0, q(a)\gamma_1(a)\phi_3, \tfrac{-d\phi_4}{da} - (b(a)+\gamma_2(a)+\mu_1(a))\phi_4, 0,0\right),$$

$$A_5 = \Big(0, (1-p(a))\sigma(a)\phi_2, (1-q(a))\gamma_1(a)\phi_3, \gamma_2(a)\phi_4, \tfrac{-d\phi_5}{da} - b(a)\phi_5, -(1-u(a))\varphi(a)\phi_6\Big),$$

$$A_6 = \left(\Psi(a)\phi_1, 0,0,0,0, \tfrac{-d\phi_6}{da} - (b(a)+\varphi(a))\phi_6\right)$$

$\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a), \phi_6(a)) \in D(A)$,

where the domain $D(A)$ is given by:

$$D(A) = \left\{\phi \in X / \phi_i \in AC[0,a_+), \phi(0) = (1,0,0,0,0,0)^T\right\},$$

and $AC[0,a_+)$ denotes the set of absolutely continuous functions on $[0,a_+)$.

We also define a non linear operator $F : X \longrightarrow X$ by:

$$(F\phi)(a) = \begin{pmatrix} -b(a) - \left((Q\phi_3)(a) + \epsilon(Q\phi_4)(a)\right)\phi_1 + \mu_1(a)\phi_4\phi_1 \\ \left((Q\phi_3)(a) + \epsilon(Q\phi_4)(a)\right)\phi_1 + \mu_1(a)\phi_4\phi_2 \\ \mu_1(a)\phi_4\phi_3 \\ \mu_1(a)\phi_4^2 \\ \mu_1(a)\phi_4\phi_5 \\ \mu_1(a)\phi_4\phi_6 \end{pmatrix}. \quad (3)$$

Here Q is a bounded linear operator on $L^1[0, a_+)$ given by

$$(Qf)(a) = k(a)\int_0^{a_+} \tilde{\beta}(a')f(a')da'.$$

Let $u(t) = (s(.,t), e(.,t), i(.,t), c(.,t), r(.,t), v(.,t))$. Thus, we can rewrite the system as an abstract Cauchy problem:

$$\frac{du(t)}{dt} = Au(t) + f(u(t)), \quad u(0) = u_0 \in X, \quad (4)$$

where $u_0(a) = (s_0(a), e_0(a), i_0(a), c_0(a), r_0(a), v_0(a))^T$.

For A and F following Inaba([9]) and Webb ([10]), we have the following results.

Lemma 1. *The operator A generates a C_0 semi-group e^{tA} and the space Ω is positively invariant with respect to the semiflow defined by e^{tA}.*

Lemma 2. *The operator F is continuously Fréchet differentiable on X.*

Theorem 3. *For each $u_0 \in X_+$, there are a maximal interval of existence $[0, t_0)$ and a unique continuous mild solution $u(t, u_0) \in X_+, t \in [0, t_0)$ for (3) such that*

$$u(t) = u_0 e^{tA} + \int_0^t e^{A(t-\tau)}F(u(\tau))d\tau. \quad (5)$$

4. DFE and Local Stability

4.1. DFE

Let consider the system (2) independent of time, we have:

$$\begin{cases} \frac{\mathrm{d}s(a)}{\mathrm{da}} &= b(a) + \varphi(a)u(a)v(a) - (-\mu_1(a)c(a) + k(a)\Lambda \\ & \qquad + \Psi(a) + b(a))s(a) \\ \frac{\mathrm{d}e(a)}{\mathrm{da}} &= k(a)\Lambda s(a) - (b(a) - \mu_1(a)c(a) + \sigma(a))e(a) \\ \frac{\mathrm{d}i(a)}{\mathrm{da}} &= \sigma(a)p(a)e(a) - (b(a) - \mu_1(a)c(a) + \gamma_1(a))i(a) \\ \frac{\mathrm{d}c(a)}{\mathrm{da}} &= q(a)\gamma_1(a)i(a) - (b(a) - \mu_1(a)c(a) + \mu_1(a) + \gamma_2(a))c(a) \\ \frac{\mathrm{d}r(a)}{\mathrm{da}} &= -(1-u(a))\varphi(a)v(a) + (1-p(a))\sigma(a)e(a) \\ & \quad + (1-q(a))\gamma_1(a)i(a) + \gamma_2(a)c(a) - (b(a) - \mu_1(a)c(a))r(a) \\ \frac{\mathrm{d}v(a)}{\mathrm{da}} &= \Psi(a)s(a) - (b(a) - \mu_1(a)c(a) + \varphi(a))v(a) \\ \Lambda &= \int_0^{a_+} \tilde{\beta}(a')(i(a') + \epsilon c(a'))\mathrm{da}' \end{cases} \tag{6}$$

With initial and boundary conditions:

$$s(0) = 1; e(0) = i(0) = c(0) = r(0) = v(0) = 0.$$

so, it's easy to see that the DFE is:

$$e^0(a) = i^0(a) = c^0(a) = r^0(a) = 0, \quad v^0(a) = 1 - s^0(a),$$

and

$$s^0(a) = e^{-\int_0^a (b(\tau)+\Psi(\tau)+\varphi(\tau)u(\tau))\mathrm{d}\tau} + \int_0^a e^{-\int_\eta^a (b(\tau)+\Psi(\tau)+\varphi(\tau)u(\tau))\mathrm{d}\tau}(b(\eta) + \varphi(\eta)u(\eta))\mathrm{d}\eta.$$

4.2. Net and Basic Reproductive Number $R(\Psi)$ and R_0

Let introduce a perturbation in systeme (2) at DFE, by setting:

$$s(a,t) = \bar{s}(a,t) + s^0(a), \quad e(a,t) = \bar{e}(a,t) + e^0(a), \quad i(a,t) = \bar{i}(a,t) + i^0(a),$$

$$c(a,t) = \bar{c}(a,t) + c^0(a), \quad r(a,t) = \bar{r}(a,t) + r^0(a), \quad v(a,t) = \bar{v}(a,t) + v^0(a)$$

And deriving these relations we obtain a new system of partial differential equations:

$$\begin{cases} \frac{\partial \bar{s}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{s}(a,t)}{\partial \mathrm{a}} &= \varphi(a)u(a)\bar{v}(a,t) - (\Psi(a)+b(a))\bar{s}(a,t) - (k(a)\bar{\lambda}(t) \\ & \qquad - \mu_1(a)\bar{c}(a,t))s^0(a) \\ \frac{\partial \bar{e}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{e}(a,t)}{\partial \mathrm{a}} &= k(a)\bar{\lambda}(t)s^0(a) - (b(a)+\sigma(a))\bar{e}(a,t) \\ \frac{\partial \bar{i}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{i}(a,t)}{\partial \mathrm{a}} &= \sigma(a)p(a)\bar{e}(a,t) - (b(a)+\gamma_1(a))\bar{i}(a,t) \\ \frac{\partial \bar{c}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{c}(a,t)}{\partial \mathrm{a}} &= q(a)\gamma_1(a)\bar{i}(a,t) - (b(a)+\mu_1(a)+\gamma_2(a))\bar{c}(a,t) \\ \frac{\partial \bar{r}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{r}(a,t)}{\partial \mathrm{a}} &= -(1-u(a))\varphi(a)\bar{v}(a,t) - (1-u(a))\varphi(a)v^0(a) \\ & \qquad + (1-p(a))\sigma(a)\bar{e}(a,t) + (1-q(a))\gamma_1(a)\bar{i}(a,t) \\ & \qquad + \gamma_2(a)\bar{c}(a,t) - b(a)\bar{r}(a,t) \\ \frac{\partial \bar{v}(a,t)}{\partial \mathrm{t}} + \frac{\partial \bar{v}(a,t)}{\partial \mathrm{a}} &= \Psi(a)\bar{s}(a,t) - (b(a)+\varphi(a))\bar{v}(a,t) \\ & \qquad + \mu_1(a)\bar{c}(a,t)v^0(a) \\ \bar{\lambda}(t) &= \int_0^{a_+} \tilde{\beta}(a')(\bar{i}(a',t) + \epsilon\bar{c}(a',t))\mathrm{d}a' \end{cases} \tag{7}$$

With initial and boundary conditions:

$$\bar{s}(0,t) = \bar{e}(0,t) = \bar{i}(0,t) = \bar{c}(0,t) = \bar{r}(0,t) = \bar{v}(0,t) = 0.$$

Let consider now the exponential solutions in the form:

$$\bar{s}(a,t) = \bar{s}(a)e^{\lambda t}; \quad \bar{e}(a,t) = \bar{e}(a)e^{\lambda t}; \quad \bar{i}(a,t) = \bar{i}(a)e^{\lambda t};$$
$$\bar{c}(a,t) = \bar{c}(a)e^{\lambda t}; \quad \bar{r}(a,t) = \bar{r}(a)e^{\lambda t} \quad \textit{and} \quad \bar{v}(a,t) = \bar{v}(a)e^{\lambda t}.$$

then we deduce:

$$
\begin{cases}
\frac{\mathrm{d}\bar{s}(a)}{\mathrm{da}} &= \varphi(a)u(a)\bar{v}(a) - (\Psi(a) + b(a) + \lambda)\bar{s}(a) - (k(a)\bar{\Lambda} - \mu_1(a)\bar{c}(a))s^0(a) \\
\frac{\mathrm{d}\bar{e}(a)}{\mathrm{da}} &= k(a)\bar{\Lambda}s^0(a) - (b(a) + \sigma(a) + \lambda)\bar{e}(a) \\
\frac{\mathrm{d}\bar{i}(a)}{\mathrm{da}} &= \sigma(a)p(a)\bar{e}(a) - (b(a) + \gamma_1(a) + \lambda)\bar{i}(a) \\
\frac{\mathrm{d}\bar{c}(a)}{\mathrm{da}} &= q(a)\gamma_1(a)\bar{i}(a) - (b(a) + \mu_1(a) + \gamma_2(a) + \lambda)\bar{c}(a) \\
\frac{\mathrm{d}\bar{r}(a)}{\mathrm{da}} &= -(1 - u(a))\varphi(a)\bar{v}(a) - (1 - u(a))\varphi(a)v^0(a)e^{-\lambda t} + (1 - p(a))\sigma(a)\bar{e}(a) \\
& +(1 - q(a))\gamma_1(a)\bar{i}(a) + \gamma_2(a)\bar{c}(a) - b(a)\bar{r}(a) - \lambda\bar{r}(a) \\
\frac{\mathrm{d}\bar{v}(a)}{\mathrm{da}} &= \Psi(a)\bar{s}(a) - (b(a) + \varphi(a) + \lambda)\bar{v}(a) + \mu_1(a)\bar{c}(a)v^0(a) \\
\bar{\Lambda} &= \int_0^{a_+} \tilde{\beta}(a')(\bar{i}(a') + \epsilon\bar{c}(a'))\mathrm{da}'
\end{cases}
\tag{8}
$$

With initial boundary condition:

$$
\bar{s}(0) = \bar{e}(0) = \bar{i}(0) = \bar{c}(0) = \bar{r}(0) = \bar{v}(0) = 0. \tag{9}
$$

From equations (8), we obtain:

$$
\bar{e}(a) = \int_0^a e^{-\int_\eta^a (\sigma(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} k(\eta)\bar{\Lambda}s^0(\eta)\mathrm{d}\eta \tag{10}
$$

$$
\bar{i}(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} \sigma(\eta)p(\eta)\bar{e}(\eta)\mathrm{d}\eta \tag{11}
$$

$$
\bar{c}(a) = \int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} q(\eta)\gamma_1(\eta)\bar{i}(\eta)\mathrm{d}\eta \tag{12}
$$

Replace (10), in expression of (11), we have:

$$
\bar{i}(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} \sigma(\eta)p(\eta) \int_0^\eta e^{-\int_\xi^\eta (\sigma(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} k(\xi)\bar{\Lambda}s^0(\xi)\mathrm{d}\xi\mathrm{d}\eta
$$

After changing the order of integration,(11) becomes:

$$
\bar{i}(a) = \bar{\Lambda} \int_0^a e^{-\int_\eta^a (b(\tau)+\lambda)\mathrm{d}\tau} k(\eta)s^0(\eta) \int_\eta^a e^{-\int_\xi^a \gamma_1(\tau)\mathrm{d}\tau} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)\mathrm{d}\xi\mathrm{d}\eta \tag{13}
$$

Substituting (13), in (12), we have:

$$\bar{c}(a) = \int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau)+b(\tau)+\lambda)\mathrm{d}\tau} q(\eta)\gamma_1(\eta)\bar{\Lambda} \int_0^\eta e^{-\int_\xi^\eta (b(\tau)+\lambda)\mathrm{d}\tau} k(\xi)s^0(\xi)$$

$$\times \int_\xi^\eta e^{-\int_\alpha^\eta \gamma_1(\tau)\mathrm{d}\tau} e^{-\int_\xi^\alpha \sigma(\tau)\mathrm{d}\tau} \sigma(\alpha)p(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta$$

After changing the order of integration twice, (12) becomes:

$$\begin{aligned}\bar{c}(a) = \bar{\Lambda} \int_0^a e^{-\int_\eta^a (b(\tau)+\lambda)\mathrm{d}\tau} k(\eta)s^0(\eta) \int_\eta^a e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \\ \times \int_\xi^a e^{-\int_\alpha^a (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta\end{aligned} \tag{14}$$

Substituting (13) and (14) in expression of Λ, we obtain:

$$\bar{\Lambda} = \int_0^{a_+} \tilde{\beta}(a')\bar{\Lambda} \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)+\lambda)\mathrm{d}\tau} k(\eta)s^0(\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)$$

$$\times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a'$$

Dividing both sides by $\bar{\Lambda} \neq 0$, we obtain a characteristic equation:

$$\begin{aligned}1 = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)+\lambda)\mathrm{d}\tau} k(\eta)s^0(\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \\ \times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a'.\end{aligned} \tag{15}$$

Let denote the right hand side of (15) by $F(\lambda)$, we define the net reproductive number (C. Castillo-Chavez and Z. Feng [4]) as $F(0) = R(\Psi)$, or explicitly as:

$$\begin{aligned}R(\Psi) = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau))\mathrm{d}\tau} k(\eta)s^0(\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \\ \times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a',\end{aligned} \tag{16}$$

with $s^0(\eta) = e^{-\int_0^\eta (b(\tau)+\Psi(\tau)+\varphi(\tau)u(\tau))\mathrm{d}\tau} + \int_0^\eta e^{-\int_\xi^\eta (b(\tau)+\Psi(\tau)+\varphi(\tau)u(\tau))\mathrm{d}\tau}(b(\xi) + \varphi(\xi)u(\xi))\mathrm{d}\xi$.

When we consider that the total population is only susceptible $\Psi(a) = 0$, and we neglect equation of $v(a)$ we found a basic reproductive number (see [4]), as $R(0) = R_0$:

$$R_0 = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau))\mathrm{d}\tau} k(\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)$$
$$\times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha)e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau}\mathrm{d}\alpha \right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a'. \tag{17}$$

4.3. Local Stability of DFE

Theorem 4. *The infection free steady-state is locally asymptotically stable if $R(\Psi) < 1$ and unstable if $R(\Psi) > 1$.*

Proof. Note that $F'(\lambda) < 0$, $\lim_{\lambda\longrightarrow+\infty} F(\lambda) = 0$, $\lim_{\lambda\longrightarrow-\infty} F(\lambda) = +\infty$. Equation (15) has a unique negative real solution λ^* if and only if $F(0) < 1$ or $R(\Psi) < 1$. Equation (15) has a unique positive (zero) real solution if $F(0) > 1$ ($F(0) = 1$) or $R(\Psi) > 1$ ($R(\Psi) = 1$).

To show that λ^* is the dominant real part of roots of $F(\lambda)$, we let $\lambda = x+iy$; $(x, y) \in \mathbb{R}^2$ be an arbitrary solution of equation (15). Note that:

$$1 = F(\lambda) = |F(x + iy)| \le F(x),$$

indicating that λ $(Re(\lambda)) \le \lambda^*$. This proove that the disease free equilibrim is locally asymptotically stable if $R(\Psi) < 1$ an unstable if $R(\Psi) > 1$. □

We may obtain a better understanding of vaccination impact by comparing the net reproductive number $R(\Psi)$ with R_0, which is called the basic reproductive number (when a purely susceptible population is considered). We can obtain an expression of R_0 in a similar case as the derivation of $R(\Psi)$ by considering the system (2) without vaccination; by assuming that $\Psi(a) = 0$ and neglecting equation of $v(a, t)$. When a reproductive number is greater than one, in the absence of vaccine; i.e $R(0) = R_0 > 1$, a vaccination programs will be used to reduce the net reproductive number $R(\Psi)$ to values below one, playing an important role to controle or eliminate the disease.

5. Existence of EE and Local Stability

In section 4 we show that the disease free equilibrium is unstable if $R(\Psi) > 1$. In fact, a non trivial steady state appears at the same times as shown below:

Theorem 5. *If $R(\Psi) > 1$, so it exists an unique endemique equilibrium of the system (2).*

Proof. The endemic equilibrium is solution of system (6) independent of time:

$$\begin{cases} \frac{\mathrm{d}s^*(a)}{\mathrm{d}a} &= b(a) + \varphi(a)u(a)v^*(a) - (-\mu_1(a)c^*(a) + k(a)\Lambda^* + \Psi(a) + b(a))s^*(a) \\ \frac{\mathrm{d}e^*(a)}{\mathrm{d}a} &= k(a)\Lambda^* s^*(a) - (b(a) - \mu_1(a)c^*(a) + \sigma(a))e^*(a) \\ \frac{\mathrm{d}i^*(a)}{\mathrm{d}a} &= \sigma(a)p(a)e^*(a) - (b(a) - \mu_1(a)c^*(a) + \gamma_1(a))i^*(a) \\ \frac{\mathrm{d}c^*(a)}{\mathrm{d}a} &= q(a)\gamma_1(a)i^*(a) - (b(a) - \mu_1(a)c^*(a) + \mu_1(a) + \gamma_2(a))c^*(a) \\ \frac{\mathrm{d}r^*(a)}{\mathrm{d}a} &= -(1-u(a))\varphi(a)v^*(a) + (1-p(a))\sigma(a)e^*(a) + (1-q(a))\gamma_1(a)i^*(a) \\ & +\gamma_2(a)c^*(a) - (b(a) - \mu_1(a)c^*(a))r^*(a) \\ \frac{\mathrm{d}v^*(a)}{\mathrm{d}a} &= \Psi(a)s^*(a) - (b(a) - \mu_1(a)c^*(a) + \varphi(a))v^*(a) \\ \Lambda^* &= \int_0^{a+} \tilde{\beta}(a')(i^*(a') + \epsilon c^*(a'))\mathrm{d}a' \end{cases} \tag{18}$$

With initial and boundary conditions:

$$s^*(0) = 1; e^*(0) = i^*(0) = c^*(0) = r^*(0) = v^*(0) = 0.$$

And the solution is given by:

$$\begin{aligned} s^*(a) &= e^{-\int_0^a (k(\tau)\Lambda^* + \Psi(\tau) + b(\tau) - \mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \\ &+ \int_0^a e^{-\int_\eta^a (k(\tau)\Lambda^* + \Psi(\tau) + b(\tau) - \mu_1(\tau)c^*(\tau))\mathrm{d}\tau}(b(\eta) + \varphi(\eta)r(\eta)v^*(\eta))\mathrm{d}\eta \end{aligned} \tag{19}$$

$$e^*(a) = \int_0^a e^{-\int_\eta^a (\sigma(\tau) + b(\tau) - \mu_1(\tau)c^*(\tau))\mathrm{d}\tau} k(\eta)\Lambda^* s^*(\eta)\mathrm{d}\eta \tag{20}$$

$$i^*(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau) + b(\tau) - \mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \sigma(\eta)p(\eta)e^*(\eta)\mathrm{d}\eta \tag{21}$$

$$c^*(a) = \int_0^a e^{-\int_\eta^a (\gamma_2(\tau)+\mu_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} q(\eta)\gamma_1(\eta)i^*(\eta)\mathrm{d}\eta \qquad (22)$$

$$r^*(a) = \int_0^a e^{-\int_\eta^a (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}(-(1-u(\eta))\varphi(\eta)v^*(\eta) + (1-p(\eta))\sigma(\eta)e^*(\eta)$$
$$+ (1-q(\eta))\gamma_1(\eta)i^*(\eta) + \gamma_2(\eta)c^*(\eta))\mathrm{d}\eta \qquad (23)$$

$$v^*(a) = \int_0^a e^{-\int_\eta^a (\varphi(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\Psi(\eta)s^*(\eta)\mathrm{d}\eta \qquad (24)$$

Substituting (20) into expression of (21), we have:

$$i^*(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\sigma(\eta)p(\eta)$$
$$\int_0^\eta e^{-\int_\xi^\eta (\sigma(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\xi)\Lambda^* s^*(\xi)\mathrm{d}\xi\mathrm{d}\eta$$

by changing the bounds of integration, we get:

$$i^*(a) = \Lambda^* \int_0^a e^{-\int_\eta^a (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\eta)s^*(\eta)$$
$$\int_\eta^a e^{-\int_\xi^a \gamma_1(\tau)\mathrm{d}\tau}e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi)\mathrm{d}\xi\mathrm{d}\eta \qquad (25)$$

Substituting(22) into expression of (25), we find:

$$c^*(a) = \int_0^a e^{-\int_\eta^a (\gamma_2(\tau)+\mu_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}q(\eta)\gamma_1(\eta)\Lambda^*$$
$$\int_0^\eta e^{-\int_\xi^\eta (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\xi)s^*(\xi)$$
$$\times \int_\xi^\eta e^{-\int_\alpha^\eta \gamma_1(\tau)\mathrm{d}\tau}e^{-\int_\xi^\alpha \sigma(\tau)\mathrm{d}\tau}\sigma(\alpha)p(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta.$$

After changing the bounds of integration twice, we obtain:

$$c^*(a) = \Lambda^* \int_0^a e^{-\int_\eta^a (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\eta)s^*(\eta)\int_\eta^a e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi)$$
$$\times \int_\xi^a e^{-\int_\alpha^a (\gamma_2(\tau)+\mu_1(\tau))\mathrm{d}\tau}q(\alpha)\gamma_1(\alpha)e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta \qquad (26)$$

At last, we substitute (25) and (26) in expression of Λ^*, and we simplify by Λ^* by considering it non zero, which give us:

$$1 = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'}(b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} k(\eta)s^*(\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'}(\gamma_2(\tau)+\mu_1(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{da}' \quad (27)$$

Let $G(\Lambda^*)$ equal to the right hand side member of equation (27), we see that an endemic equilibrium exists if equation (27) admits a positive solution. For $\Lambda^* = 0$, we obtain $s^*(a)$ equal to $s^0(a)$, so $G(0) = R(\Psi)$, as soon as $G(0) > 1$. We know that $s^*(a) + e^*(a) + i^*(a) + c^*(a) + v^*(a) = 1$, which leads that $i^*(a) + \epsilon c^*(a) < 1$. Therefore Λ^* nonzero, and from (27), we gets:

$$G(\Lambda^*) = \frac{1}{\Lambda^*} \int_0^{a_+} \tilde{\beta}(a')(i^*(a') + \epsilon c^*(a'))\mathrm{da}' < \frac{1}{\Lambda^*} \int_0^{a_+} \tilde{\beta}(a')\mathrm{da}' \quad (28)$$

The right hand side of (28) goes to zero when Λ^* goes to infinity. However $G(\Lambda^*) = 1$ admits an unique solution in interval $[0, +\infty[$. Moreover, if $\Lambda^* > \int_0^{a_+} \tilde{\beta}(a')\mathrm{da}'$, then $G(\Lambda^*) < 1$. Hence the equation (27) admits an unique solution in interval $\left[0, \int_0^{a_+} \tilde{\beta}(a')\mathrm{da}'\right]$ and the endemic equilibrium exists. □

Now linearize the system (2) at endemic equilibrium, which gives by posing:

$$s(a,t) = \hat{s}(a,t) + s^*(a), \quad e(a,t) = \hat{e}(a,t) + e^*(a), \quad i(a,t) = \hat{i}(a,t) + i^*(a),$$

$$c(a,t) = \hat{c}(a,t) + c^*(a), \quad r(a,t) = \hat{r}(a,t) + r^*(a) \quad v(a,t) = \hat{v}(a,t) + v^*(a)$$

by deriving these equations, we obtain a new system of partial differential equation:

$$
\begin{cases}
\frac{\partial \hat{s}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{s}(a,t)}{\partial \mathrm{a}} = \varphi(a)u(a)\hat{v}(a,t) - (\Psi(a) + b(a) + k(a)\Lambda^* \\
\qquad - \mu_1(a)c^*(a))\hat{s}(a,t) - (\hat{\lambda}(a,t) - \mu_1(a)\hat{c}(a,t))s^*(a) \\
\frac{\partial \hat{e}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{e}(a,t)}{\partial \mathrm{a}} = \hat{\lambda}(a,t)s^*(a) + k(a)\Lambda^*\hat{s}(a,t) \\
\qquad - (b(a) + \sigma(a) - \mu_1(a)c^*(a))\hat{e}(a,t) + \mu_1(a)\hat{c}(a,t)e^*(a) \\
\frac{\partial \hat{i}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{i}(a,t)}{\partial \mathrm{a}} = \sigma(a)p(a)\hat{e}(a,t) - (b(a) \\
\qquad + \gamma_1(a) - \mu_1(a)c^*(a))\hat{i}(a,t) + \mu_1(a)\hat{c}(a,t)i^*(a) \\
\frac{\partial \hat{c}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{c}(a,t)}{\partial \mathrm{a}} = q(a)\gamma_1(a)\hat{i}(a,t) - (b(a) + \mu_1(a) + \gamma_2(a) \\
\qquad - \mu_1(a)c^*(a))\hat{c}(a,t) + \mu_1(a)\hat{c}(a,t)c^*(a) \\
\frac{\partial \hat{r}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{r}(a,t)}{\partial \mathrm{a}} = -(1 - u(a))\varphi(a)\hat{v}(a,t) + (1 - p(a))\sigma(a)\hat{e}(a,t) \\
\qquad + (1 - q(a))\gamma_1(a)\hat{i}(a,t) + \gamma_2(a)\hat{c}(a,t) - b(a)\hat{r}(a,t) \\
\qquad + \mu_1(a)\hat{c}(a,t)v^*(a) + \mu_1(a)c^*(a)\hat{v}(a,t) \\
\frac{\partial \hat{v}(a,t)}{\partial \mathrm{t}} + \frac{\partial \hat{v}(a,t)}{\partial \mathrm{a}} = \Psi(a)\hat{s}(a,t) - (b(a) + \varphi(a) - \mu_1(a)c^*(a))\hat{v}(a,t) \\
\qquad + \mu_1(a)\hat{c}(a,t)v^*(a) \\
\hat{\lambda}(a,t) = k(a)\int_0^{a_+} \tilde{\beta}(a')(\hat{i}(a',t) + \epsilon\hat{c}(a',t))\mathrm{d}a'
\end{cases}
\tag{29}
$$

with initial and boundary conditions:

$$\hat{s}(0,t) = \hat{e}(0,t) = \hat{i}(0,t) = \hat{c}(0,t) = \hat{r}(0,t) = \hat{v}(0,t) = 0.$$

Now considering an exponential solution in the form:

$$\hat{s}(a,t) = \tilde{s}(a)e^{\lambda t}, \quad \hat{e}(a,t) = \tilde{e}(a)e^{\lambda t}, \quad \hat{i}(a,t) = \tilde{i}(a)e^{\lambda t}, \hat{c}(a,t) = \tilde{c}(a)e^{\lambda t},$$

$$\hat{r}(a,t) = \tilde{r}(a)e^{\lambda t} \quad \hat{v}(a,t) = \tilde{v}(a)e^{\lambda t}, \quad \hat{\lambda}(a,t) = k(a)\tilde{\lambda}e^{\lambda t}$$

We get:

$$\begin{cases} \frac{\mathrm{d}\tilde{s}(a)}{\mathrm{da}} = \varphi(a)r(a)\tilde{v}(a) - (\Psi(a) + b(a) + k(a)\Lambda^* - \mu_1(a)c^*(a))\tilde{s}(a) \\ \qquad\qquad - (k(a)\tilde{\lambda} - \mu_1(a)\tilde{c}(a))s^*(a) - \lambda\tilde{s}(a) \\ \frac{\mathrm{d}\tilde{e}(a)}{\mathrm{da}} = k(a)\tilde{\lambda}s^*(a) + k(a)\Lambda^*\tilde{s}(a) - (b(a) + \sigma(a) \\ \qquad\qquad - \mu_1(a)c^*(a))\tilde{e}(a) + \mu_1(a)\tilde{c}(a)e^*(a) - \lambda\tilde{e}(a) \\ \frac{\mathrm{d}\tilde{i}(a)}{\mathrm{da}} = \sigma(a)p(a)\tilde{e}(a) - (b(a) + \gamma_1(a) - \mu_1(a)c^*(a))\tilde{i}(a) \\ \qquad\qquad + \mu_1(a)\tilde{c}(a)i^*(a) - \lambda\tilde{i}(a) \\ \frac{\mathrm{d}\tilde{c}(a)}{\mathrm{da}} = q(a)\gamma_1(a)\tilde{i}(a) - (b(a) + \mu_1(a) + \gamma_2(a) - \mu_1(a)c^*(a))\tilde{c}(a) \\ \qquad\qquad + \mu_1(a)\tilde{c}(a)c^*(a) - \lambda\tilde{c}(a) \\ \frac{\mathrm{d}\tilde{r}(a)}{\mathrm{da}} = -(1 - u(a))\varphi(a)\tilde{v}(a) + (1 - p(a))\sigma(a)\tilde{e}(a) + (1 - q(a))\gamma_1(a)\tilde{i}(a) \\ \qquad + \gamma_2(a)\tilde{c}(a) - b(a)\tilde{r}(a) + \mu_1(a)\tilde{c}(a)v^*(a) + \mu_1(a)c^*(a)\tilde{v}(a) - \lambda\tilde{r}(a) \\ \frac{\mathrm{d}\tilde{v}(a)}{\mathrm{da}} = \Psi(a)\tilde{s}(a) - (b(a) + \varphi(a) - \mu_1(a)c^*(a))\tilde{v}(a) \\ \qquad\qquad + \mu_1(a)\tilde{c}(a)v^*(a) - \lambda\tilde{v}(a) \\ \tilde{\lambda} = \int_0^{a_+} \tilde{\beta}(a')(\tilde{i}(a') + \epsilon\tilde{c}(a'))\mathrm{da}' \end{cases}$$

with initial boundary values:

$$\tilde{s}(0) = \tilde{e}(0) = \tilde{i}(0) = \tilde{c}(0) = \tilde{r}(0) = \tilde{v}(0) = 0.$$

Let:

$$\bar{s}(a) = \frac{\tilde{s}(a)}{\tilde{\lambda}}, \quad \bar{e}(a) = \frac{\tilde{e}(a)}{\tilde{\lambda}}, \quad \bar{i}(a) = \frac{\tilde{i}(a)}{\tilde{\lambda}},$$
$$\bar{c}(a) = \frac{\tilde{c}(a)}{\tilde{\lambda}}, \quad \bar{r}(a) = \frac{\tilde{r}(a)}{\tilde{\lambda}}, \quad \bar{v}(a) = \frac{\tilde{v}(a)}{\tilde{\lambda}}$$

which gives us:

$$\begin{cases} \frac{\mathrm{d}\bar{s}(a)}{\mathrm{da}} = \varphi(a)r(a)\bar{v}(a) - (\Psi(a) + b(a) + k(a)\Lambda^* - \mu_1(a)c^*(a))\bar{s}(a) \\ \qquad - (k(a) - \mu_1(a)\bar{c}(a))s^*(a) - \lambda\bar{s}(a) \\ \frac{\mathrm{d}\bar{e}(a)}{\mathrm{da}} = k(a)s^*(a) + k(a)\Lambda^*\bar{s}(a) - (b(a) + \sigma(a) - \mu_1(a)c^*(a))\bar{e}(a) \\ \qquad + \mu_1(a)\bar{c}(a)e^*(a) - \lambda\bar{e}(a) \\ \frac{\mathrm{d}\bar{i}(a)}{\mathrm{da}} = \sigma(a)p(a)\bar{e}(a) - (b(a) + \gamma_1(a) - \mu_1(a)c^*(a))\bar{i}(a) \\ \qquad + \mu_1(a)\bar{c}(a)i^*(a) - \lambda\bar{i}(a) \\ \frac{\mathrm{d}\bar{c}(a)}{\mathrm{da}} = q(a)\gamma_1(a)\bar{i}(a) - (b(a) + \mu_1(a) + \gamma_2(a) - \mu_1(a)c^*(a))\bar{c}(a) \\ \qquad + \mu_1(a)\bar{c}(a)c^*(a) - \lambda\bar{c}(a) \\ \frac{\mathrm{d}\bar{r}(a)}{\mathrm{da}} = -(1 - u(a))\varphi(a)\bar{v}(a) + (1 - p(a))\sigma(a)\bar{e}(a) \\ \quad + (1 - q(a))\gamma_1(a)\bar{i}(a) + \gamma_2(a)\bar{c}(a) - b(a)\bar{r}(a) + \mu_1(a)\bar{c}(a)v^*(a) \\ \qquad + \mu_1(a)c^*(a)\bar{v}(a) - \lambda\bar{r}(a) \\ \frac{\mathrm{d}\bar{v}(a)}{\mathrm{da}} = \Psi(a)\bar{s}(a) - (b(a) + \varphi(a) - \mu_1(a)c^*(a))\bar{v}(a) \\ \qquad + \mu_1(a)\bar{c}(a)v^*(a) - \lambda\bar{v}(a) \\ 1 = \int_0^{a_+} \tilde{\beta}(a')(\bar{i}(a') + \epsilon\bar{c}(a'))\mathrm{da}' \end{cases} \tag{30}$$

with initial boundary values:

$$\bar{s}(0) = \bar{e}(0) = \bar{i}(0) = \bar{c}(0) = \bar{r}(0) = \bar{v}(0) = 0.$$

And the solution of (30) satisfy equation:

$$1 = \int_0^{a_+} \tilde{\beta}(a')(\bar{i}(a') + \epsilon\bar{c}(a'))\mathrm{da}' \tag{31}$$

Let $H(\lambda)$ be the right hand side of (31). Which satisfy the next proposition:

Proposition 6. *1. $H(\lambda)$ is a decreasing function in λ which tends to 0, when $\lambda \longrightarrow +\infty$,*

2. $H(0) < 1$.

Proof. From the system (30), we have:

$$\bar{s}(a) = \int_0^a e^{-\int_\eta^a (k(\tau)\Lambda^* + \Psi(\tau) + b(\tau) - \mu_1(\tau)c^*(\tau) + \lambda)\mathrm{d}\tau} (\varphi(\eta)r(\eta)\bar{v}(\eta) - k(\eta)s^*(\eta) + \mu_1(\eta)\bar{c}(\eta)s^*(\eta))\mathrm{d}\eta \tag{32}$$

$$\bar{e}(a) = \int_0^a e^{-\int_\eta^a (\sigma(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau} (k(\eta)s^*(\eta) + k(\eta)\Lambda^*\bar{s}(\eta) + \mu_1(\eta)\bar{c}(\eta)e^*(\eta))\mathrm{d}\eta \quad (33)$$

$$\bar{i}(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau} (\sigma(\eta)p(\eta)\bar{e}(\eta) + \mu_1(\eta)\bar{c}(\eta)i^*(\eta))\mathrm{d}\eta \quad (34)$$

$$\bar{c}(a) = \int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau} (q(\eta)\gamma_1(\eta)\bar{i}(\eta) + \mu_1(\eta)\bar{c}(\eta)c^*(\eta))\mathrm{d}\eta \quad (35)$$

$$\begin{aligned}\bar{r}(a) = \int_0^a e^{-\int_\eta^a (b(\tau)+\lambda)\mathrm{d}\tau} (&-(1-u(\eta))\varphi(\eta)\bar{v}(\eta) \\ &+ (1-p(\eta))\sigma(\eta)\bar{e}(\eta) + (1-q(\eta))\gamma_1(\eta)\bar{i}(\eta) \\ &+ \gamma_2(\eta)\bar{c}(\eta) + \mu_1(\eta)\bar{c}(\eta)v^*(\eta) + \mu_1(\eta)c^*(\eta)\bar{v}(\eta))\mathrm{d}\eta \quad (36)\end{aligned}$$

$$\bar{v}(a) = \int_0^a e^{-\int_\eta^a (\varphi(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau} (\Psi(\eta)\bar{s}(\eta) + \mu_1(\eta)\bar{c}(\eta)v^*(\eta))\mathrm{d}\eta \quad (37)$$

By substuting (21) in (34), we get:

$$\begin{aligned}\bar{i}(a) &= \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau} \sigma(\eta)p(\eta)\bar{e}(\eta)\mathrm{d}\eta \\ &+ \int_0^a e^{-\lambda(a-\eta)}\mu_1(\eta)\bar{c}(\eta) \int_0^\eta e^{-\int_\xi^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \sigma(\xi)p(\xi)e^*(\xi)\mathrm{d}\xi\mathrm{d}\eta \quad (38)\end{aligned}$$

By substuting (21), (22) and (38) in (36), we obtain:

$$\begin{aligned}
\bar{c}(a) = &\int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\eta)\gamma_1(\eta) \\
&\int_0^\eta e^{-\int_\xi^a (b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} e^{-\int_\xi^\eta \gamma_1(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)\bar{e}(\xi)\mathrm{d}\xi\mathrm{d}\eta + \\
&\int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau)-\gamma_1(\tau))\mathrm{d}\tau} q(\eta)\gamma_1(\eta) \int_0^\eta e^{-\lambda(a-\xi)}\mu_1(\xi)\bar{c}(\xi) \\
&\int_0^\xi e^{-\int_\alpha^a (b(\tau)+\gamma_1(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \\
&\times \sigma(\alpha)p(\alpha)e^*(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta \\
&+ \int_0^a e^{-\lambda(a-\eta)}\mu_1(\eta)\bar{c}(\eta) \int_0^\eta e^{-\int_\xi^a (\mu_1(\tau)+\gamma_2(\tau)-\gamma_1(\tau))\mathrm{d}\tau} q(\xi)\gamma_1(\xi) \\
&\times \int_0^\xi e^{-\int_\alpha^a (b(\tau)+\gamma_1(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \sigma(\alpha)p(\alpha)e^*(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta
\end{aligned}$$

By changing the bounds of integration in the second and third integrale, we obtain:

$$\begin{aligned}
\bar{c}(a) = &\int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\eta)\gamma_1(\eta) \\
&\int_0^\eta e^{-\int_\xi^a (b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} e^{-\int_\xi^\eta \gamma_1(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)\bar{e}(\xi)\mathrm{d}\xi\mathrm{d}\eta \\
&+ \int_0^a e^{-\int_\eta^a (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\eta)\gamma_1(\eta) \\
&\int_0^\eta e^{-\int_\xi^a (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} e^{-\int_\xi^\eta \gamma_1(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)e^*(\xi) \\
&\times \int_\xi^a e^{-\lambda(a-\alpha)}\mu_1(\alpha)\bar{c}(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta
\end{aligned} \tag{39}$$

Now, let take (38), and change the bounds of integration for the second integrale, we find:

$$\bar{i}(a) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau)+\lambda)\mathrm{d}\tau}\sigma(\eta)p(\eta)\bar{e}(\eta)\mathrm{d}\eta$$
$$+ \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\sigma(\eta)p(\eta)e^*(\eta) \tag{40}$$
$$\int_\eta^a e^{-\lambda(a-\xi)}\mu_1(\xi)\bar{c}(\xi)\mathrm{d}\xi\mathrm{d}\eta$$

So, (31) becomes:

$$H(\lambda) = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}$$
$$\Big[e^{-\int_\eta^{a'} \gamma_1(\tau)\mathrm{d}\tau}\sigma(\eta)p(\eta)\bar{e}(\eta) + \epsilon e^{-\int_\eta^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau}$$
$$\times\ q(\eta)\gamma_1(\eta)\int_0^\eta e^{-\int_\xi^\eta (b(\tau)+\lambda+\gamma_1(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\sigma(\xi)p(\xi)\bar{e}(\xi)\mathrm{d}\xi\Big]\,\mathrm{d}\eta\mathrm{d}a'$$
$$+ \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}$$
$$\Big[e^{-\int_\eta^{a'} \gamma_1(\tau)\mathrm{d}\tau}\sigma(\eta)p(\eta)e^*(\eta)\int_\eta^{a'} e^{-\lambda(a'-\xi)}\mu_1(\xi)\bar{c}(\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a'$$
$$+\epsilon e^{-\int_\eta^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau}q(\eta)\gamma_1(\eta)\int_0^\eta e^{-\int_\xi^\eta (b(\tau)+\gamma_1(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\sigma(\xi)p(\xi)e^*(\xi)$$
$$\int_\xi^{a'} e^{-\lambda(a'-\alpha)}\ \times\mu_1(\alpha)\bar{c}(\alpha)\mathrm{d}\alpha\mathrm{d}\xi\big]\,\mathrm{d}\eta\mathrm{d}a' \tag{41}$$

From equation (31), we have $\bar{i}(a) + \epsilon\bar{c}(a) > 0$, which leads that $\bar{e}(a) > 0$. However, from (41) we have $H(\lambda)$ which decrease exponentially and $H(\lambda)$ goes to 0, when λ goes to $+\infty$. By substituting (20) in (33) and all in (41), we obtain:

$$H(0) = \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} k(\eta)s^*(\eta) \int_\eta^{a'} e^{-\int_\eta^{\xi} \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)$$

$$\times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^{\alpha} \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}a'$$

$$+ \Lambda^* \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} k(\eta)\bar{s}(\eta) \int_\eta^{a'} e^{-\int_\eta^{\xi} \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi)$$

$$\times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^{\alpha} \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}a'$$

$$+ \Lambda^* \int_0^{a_+} \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} k(\eta)s^*(\eta) \int_\eta^{a'} e^{-\int_\eta^{\xi} \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \times$$

$$\left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} \int_\eta^{a'} \mu_1(\alpha)\bar{c}(\alpha)\mathrm{d}\alpha + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha) e^{-\int_\xi^{\alpha} \gamma_1(\tau)\mathrm{d}\tau} \right.$$

$$\left. \times \int_\alpha^{a'} \mu_1(\theta)\bar{c}(\theta)\mathrm{d}\theta \mathrm{d}\alpha \right] \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}a' \tag{42}$$

The first integrale of equation (41) is equal to one. On other hand let

$$y(a) = \bar{s}(a) + \bar{e}(a) + \bar{i}(a) + \bar{c}(a) + \bar{v}(a).$$

From (30) we have:

$$\frac{\mathrm{d}y(a)}{\mathrm{d}a} = -\varphi(a)\bar{v}(a)(1-r(a)) - \sigma(a)\bar{e}(a)(1-p(a)) - \gamma_1(a)\bar{i}(a)(1-q(a)) - \gamma_2(a)\bar{c}(a) - (b(a)+\lambda-\mu_1(a)c^*(a))y(a) \tag{43}$$

which have solution:

$$y(a) = -\int_0^a e^{-\int_\eta^a (b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau} \left[\varphi(\eta)\bar{v}(\eta)(1-r(\eta)) + \sigma(\eta)\bar{e}(\eta)(1-p(\eta)) + \gamma_1(\eta)\bar{i}(\eta)(1-q(\eta)) - \gamma_2(\eta)\bar{c}(\eta) \right] \mathrm{d}\eta \tag{44}$$

As $\bar{i}(a) + \epsilon\bar{c}(a) > 0$ and the equation (35) we have $\bar{i}(a) > 0$ et $\bar{c}(a) > 0$. And equation (44) show that $y(a) < 0$, also from (34) $\bar{e}(a) > 0$. So $\bar{s}(a) + \bar{v}(a) < 0$. On other hand equation (37) shows that $\bar{s}(a)$ and $\bar{v}(a)$ have the same sign. Now replace (24) in (35), we gets:

$$\begin{aligned}\bar{v}(a) = &\int_0^a e^{-\int_\eta^a(\varphi(\tau)+b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\Psi(\eta)\bar{s}(\eta)\mathrm{d}\eta\\ &+\int_0^a e^{-\int_\eta^a(\varphi(\tau)+b(\tau)+\lambda-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\mu_1(\eta)\bar{c}(\eta)\\ &\times\int_0^\eta e^{-\int_\xi^\eta(\varphi(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\Psi(\xi)s^*(\xi)\mathrm{d}\xi\mathrm{d}\eta\end{aligned}$$

After changing the bounds of integration, we obtain:

$$\begin{aligned}\bar{v}(a) = &\int_0^a e^{-\int_\eta^a(\varphi(\tau)+b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}\Psi(\eta)\\ &\left[e^{-\lambda(a-\eta)}\bar{s}(\eta)+s^*(\eta)\int_\eta^a e^{-\lambda(a-\xi)}\mu_1(\xi)\bar{c}(\xi)\mathrm{d}\xi\right]\mathrm{d}\eta\end{aligned} \tag{45}$$

expression of (45) is negative, this shows that:

$$e^{-\lambda(a-\eta)}\bar{s}(\eta)+s^*(\eta)\int_\eta^a e^{-\lambda(a-\xi)}\mu_1(\xi)\bar{c}(\xi)\mathrm{d}\xi < 0$$

In particular for $\lambda = 0$, we have:

$$\bar{s}(\eta)+s^*(\eta)\int_\eta^a \mu_1(\xi)\bar{c}(\xi)\mathrm{d}\xi < 0 \tag{46}$$

Hence the other two integrales of (42) gives:

$$\begin{aligned}&\Lambda^*\int_0^{a_+}\tilde{\beta}(a')\int_0^{a'} e^{-\int_\eta^{a'}(b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\eta)\\ &\int_\eta^{a'} e^{-\int_\eta^\xi\sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi)e^{-\int_\xi^{a'}\gamma_1(\tau)\mathrm{d}\tau}\left[\bar{s}(\eta)+s^*(\eta)\times\int_\eta^a\mu_1(\alpha)\bar{c}(\alpha)\mathrm{d}\alpha\right]\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a'\\ &+\epsilon\Lambda^*\int_0^{a_+}\tilde{\beta}(a')\int_0^{a'} e^{-\int_\eta^{a'}(b(\tau)-\mu_1(\tau)c^*(\tau))\mathrm{d}\tau}k(\eta)\int_\eta^{a'} e^{-\int_\eta^\xi\sigma(\tau)\mathrm{d}\tau}\\ &\sigma(\xi)p(\xi)\int_\xi^{a'} e^{-\int_\alpha^{a'}(\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau}q(\alpha)\gamma_1(\alpha)e^{-\int_\xi^\alpha\gamma_1(\tau)\mathrm{d}\tau}e^{-\int_\xi^{a'}\gamma_1(\tau)\mathrm{d}\tau}\end{aligned}$$

$$\left[\bar{s}(\eta) + s^*(\eta)\int_\alpha^a \mu_1(\theta)\bar{c}(\theta)\mathrm{d}\theta\right] \times \mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}a' < 0$$

Which gives us $H(0) < 1$. □

Consequently $H(\lambda) = 1$, admits an unique real solution which is negative, and all the complex solution have real parts small than the unique real solution. This give us this next result on stability of endemic equilibrium.

Theorem 7. *If $R_0 > 1$, The endemic equilibrium of (2) is stable.*

6. Global Stability of DFE

Theorem 8. *If $R_0 < 1$, the disease free equilibrium (2) is globally asymptotically stable.*

Proof. For the global stability of disease free equilibrium, we will show that $s(a,t) \longrightarrow s^0(a)$, $e(a,t) \longrightarrow 0$, $i(a,t) \longrightarrow 0$, $c(a,t) \longrightarrow 0$, $r(a,t) \longrightarrow 0$, $v(a,t) \longrightarrow 1 - s^0(a)$ when $t \longrightarrow +\infty$.

Using the method of characteristic curves in the second, third and fourth equation of the system (2), we get:

$$e(a,t) = \int_0^a e^{-\int_\xi^a (\sigma(\tau)+b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau} k(\xi)\lambda(t-a+\xi)s(\xi,t-a+\xi))\mathrm{d}\xi, \quad \forall \quad t \geq a \tag{47}$$

$$i(a,t) = \int_0^a e^{-\int_\eta^a (\gamma_1(\tau)+b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau} \sigma(\eta)p(\eta)e(\eta,t-a+\eta))\mathrm{d}\eta, \quad \forall \quad t \geq a \tag{48}$$

$$c(a,t) = \int_0^a e^{-\int_\alpha^a (\gamma_2(\tau)+\mu_1(\tau)+b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau} q(\alpha)\gamma_1(\alpha)i(\alpha,t-a+\alpha))\mathrm{d}\alpha, \quad \forall \quad t \geq a \tag{49}$$

Substituting (47) in (48), we have:

$$i(a,t)=\int_0^a e^{-\int_\eta^a \gamma_1(\tau)\mathrm{d}\tau}\sigma(\eta)p(\eta)\int_0^\eta e^{-\int_\xi^a (b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau} \times k(\xi)\lambda(t-a+\xi)s(\xi,t-a+\xi)e^{-\int_\xi^\eta \sigma(\tau)\mathrm{d}\tau}\mathrm{d}\xi\mathrm{d}\eta$$

After changing of bounds of integration, we found:

$$i(a,t)=\int_0^a e^{-\int_\eta^a (b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau}k(\eta)\lambda(t-a+\eta) \times s(\eta,t-a+\eta)\int_\eta^a e^{-\int_\xi^a \gamma_1(\tau)\mathrm{d}\tau}e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi)\mathrm{d}\xi\mathrm{d}\eta \quad (50)$$

Now substituting (50) in (49), we obtain:

$$c(a,t)=\int_0^a e^{-\int_\alpha^a (\gamma_2(\tau)+\mu_1(\tau)+b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau}q(\alpha)\gamma_1(\alpha) \int_0^\alpha e^{-\int_\eta^\alpha (b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau} \times k(\eta)\lambda(t-a+\eta)s(\eta,t-a+\eta)\int_\eta^\alpha e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau}e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\alpha$$

After changing of bounds of integration twice, and using $\alpha=\eta$, $\eta=\xi$, et $\xi=\alpha$ we get:

$$c(a,t)=\int_0^a e^{-\int_\eta^a (b(\tau)-\mu_1(\tau)c(\tau,t-a+\tau))\mathrm{d}\tau}k(\eta)\lambda(t-a+\eta)s(\eta,t-a+\eta) \int_\eta^a e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau}\sigma(\xi)p(\xi) \times\int_\xi^a e^{-\int_\alpha^a (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau}e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\eta \quad (51)$$

By injecting the value of (50) and (51) in the expression of $\lambda(t)$ in the system (2), then:

$$\lambda(t)=\int_0^t \tilde\beta(a')(i(a',t)+\epsilon c(a',t))\mathrm{d}a'+\int_t^{a_+}\tilde\beta(a')(i(a',t)+\epsilon c(a',t))\mathrm{d}a'$$

$$\begin{aligned}\lambda(t) = & \int_0^t \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c(\tau,t-a'+\tau))\mathrm{d}\tau} k(\eta)\lambda(t-a'+\eta)s(\eta,t-a'+\eta) \\ & \times \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \\ & \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi \mathrm{d}\eta \mathrm{da}' \\ & + \int_t^{a_+} \tilde{\beta}(a')(i(a',t)+\epsilon c(a',t))\mathrm{da}'. \end{aligned} \tag{52}$$

Since $e(a,t)$ and $i(a,t)$ do not exceed one, and same is true for their sum, the last integral can be estimated by $\int_t^{a_+} \tilde{\beta}(a')\mathrm{da}'$ which decreases ton zero when $t \longrightarrow +\infty$. Note that $s(a,t) \le 1$ and $e^{\mu_1(\tau)c(\tau,t-a+\tau)\mathrm{d}\tau} < e$, so:

$$\begin{aligned} & \lambda(t) \le \\ & \int_0^t \tilde{\beta}(a') \int_0^{a'} e^{-\int_\eta^{a'} (b(\tau)-\mu_1(\tau)c(\tau,t-a'+\tau))\mathrm{d}\tau} k(\eta)\lambda(t-a'+\eta) \int_\eta^{a'} e^{-\int_\eta^\xi \sigma(\tau)\mathrm{d}\tau} \sigma(\xi)p(\xi) \\ & \quad \times \left[e^{-\int_\xi^{a'} \gamma_1(\tau)\mathrm{d}\tau} + \epsilon \int_\xi^{a'} e^{-\int_\alpha^{a'} (\mu_1(\tau)+\gamma_2(\tau))\mathrm{d}\tau} e^{-\int_\xi^\alpha \gamma_1(\tau)\mathrm{d}\tau} \mathrm{d}\alpha \right] \mathrm{d}\xi \mathrm{d}\eta \mathrm{da}' \\ & \qquad + \int_t^{a_+} \tilde{\beta}(a')\mathrm{da}'. \end{aligned}$$

Taking the limit supremum when $t \longrightarrow +\infty$ in both side of the inequation and using Fatou's lemma, we obtain:

$$\lim_{t \longrightarrow +\infty} \lambda(t) \le R_0 \limsup_{t \longrightarrow +\infty} \lambda(t). \tag{53}$$

As we suppose that $R_0 < 1$, inequality (53) is true if

$$\limsup_{t \longrightarrow +\infty} \lambda(t) = 0.$$

By using the result of (47), (48) et (49), we see that:

$$\limsup_{t \longrightarrow +\infty} e(a,t) = 0, \quad \limsup_{t \longrightarrow +\infty} i(a,t) = 0, \quad \limsup_{t \longrightarrow +\infty} c(a,t) = 0,$$

pointwise in a. Furthermore from equations of the system (2) and in fact that the sum of $s(a,t), e(a,t), i(a,t), c(a,t), r(a,t)$ et $v(a,t)$ add equal to 1, it is clear that:

$$\limsup_{t \longrightarrow +\infty} s(a,t) = s^0(a), \quad \limsup_{t \longrightarrow +\infty} v(a,t) = 1 - s^0(a).$$

Therefore the disease free equilibrium is globally asymptotically stable if $R_0 < 1$. □

The theorem show that there is no endemic equilibrium for parameters such that $R_0 < 1$. But, it was not shown that endemic equilibrium exist for $R(\Psi) < 1$ et $R_0 > 1$. It is possible there exist a bifurcation at endemic equilibrium for certains parameters values satisfying $R(\Psi) < 1 < R_0$. The global stability of disease free equilibrium implies that, giving a set of parameters satisfying $R_0 < 1$, for any initial positive number of infectious individuals, the number of latent infectious individuals and the normal infectious individuals decreases to zero. A such prediction can be of important use in planning and evaluation of disease control policy.

7. Simulation and Conclusion

The parameters used are essentially information we obtain in Niger Republic in the center of immunization, in the National Hospital and in the National Institute of Statistics. The parameters are listed below:

$b(a) = 0.0461$	$\Psi(a) = 0.032$	$\varphi(a) = 0.0016$	$\mu(a) = 0.0116$
$\mu_1(a) = 0.004$	$\gamma_1(a) = 0.08$	$\lambda(a,t) = 0.2$	

We simulate the latently, infectious and carriers functions in the case where $R_0 < 1$, witch give us Figures 2-4.

In Figure 2, we see that, the age to become latently is between $0 - 5$ years, and is more than other classe of ages.

For Infectious, in Figure 3 we see that the age of infection is after 4 years. But this become very important after 30 years.

And the Figure 4, show that the age to become chroniques is after 40 years.

We have shown that the desease is contracted at our younger age, and the person who contacted HBV become carrier around forty years, this affected the economy of a country in endemic area like Niger Republic. We see also that the additionnal mortality related to HBV affect carrier person. For next research, we will see the impact of vertical transmission in this model and best vaccination strategy in endemic area.

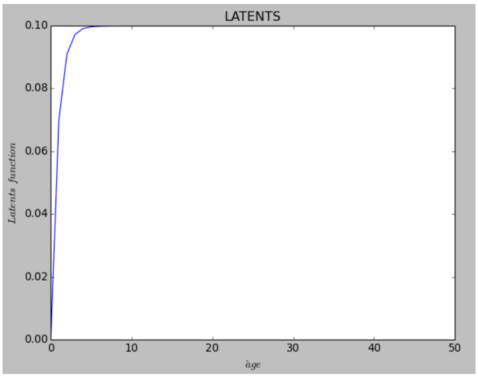

Figure 2: Latently function

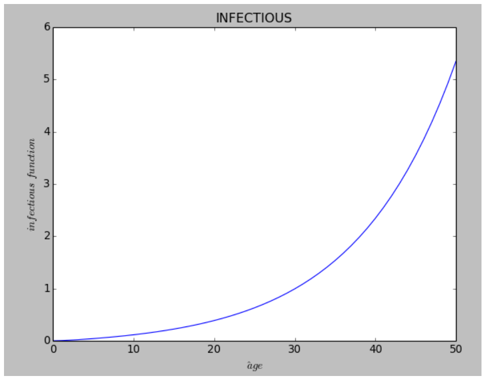

Figure 3: Infectious function

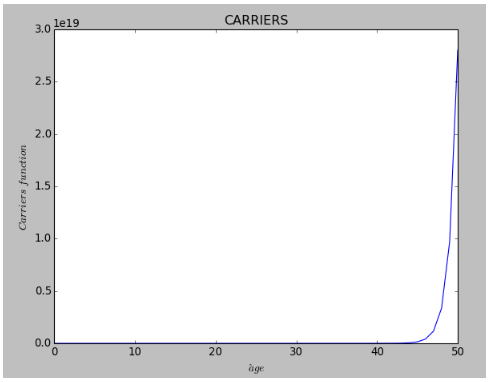

Figure 4: Carriers function

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