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MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR EULER-LAGRANGE EQUATION

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Abstract: This paper deals with the existence of multiplicity solutions for the following quasilinear Euler-Lagrange equation

$$-\,div\big((a(x)+\mid u\mid^{\gamma})\mid \nabla u\mid^{p-2}\nabla u\big)+\frac{\gamma}{p}\mid u\mid^{\gamma-2}u\mid \nabla u\mid^{p}$$
$$=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad \text{in } \Omega,$$

with zero Dirichlet boundary condition, under the assumption $1<\theta<p<q<\frac{p*}{p}(\gamma+p)$ and $\gamma>1$.

By using critical point methods we obtain the existence of multiplicity solutions for the above equation.

Key Words: Euler-Lagrange equation, weak slution, truncated function, nonsmooth critical point theory

1. Introduction

In this paper we study the following equation.

$$-\,div((a(x)+\mid u\mid^{\gamma})\mid \nabla u\mid^{p-2}\nabla u)+\frac{\gamma}{p}\mid u\mid^{\gamma-2}u\mid \nabla u\mid^{p}$$
$$=\lambda\mid u\mid^{\theta-2}u+\mid u\mid^{q-2}u\quad \text{in } \Omega,\quad (1.1)$$

with zero Dirichlet boundary condition

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$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

In this case, the functional corresponding to the quasilinear Euler-Lagrange equation J is

$$J(u) = \frac{1}{p}\int_{\Omega}(a(x)+ \mid u \mid^{\gamma}) \mid \nabla u \mid^{p} - \frac{\lambda}{\theta}\int_{\Omega} \mid u \mid^{\theta} - \frac{1}{q}\int_{\Omega} \mid u \mid^{q}, \tag{1.3}$$

where $\gamma > 1$, Ω is a bounded,open subset of R^N with $N > 2$, $1 < p < N$ and a(x) is a measurable function such that for two constants α and β

$$0 < \alpha \leq a(x) \leq \beta \quad \text{a.e. } x \in \Omega \tag{1.4}$$

During this work,we focus on the case $1 < \theta < p < q < \frac{p*}{p}(\gamma + p)$. The functional J is not Gâteau-differentiable in $W_0^{1,p}(\Omega)$ but is only differentiable through the direction of $W_0^{1,p}(\Omega) \cap Ł^{\infty}(\Omega)$.

The main difficulty is due to the term $\mid u \mid^{\gamma}$ which,although we assume that $1 < \theta < p < q < p^*$ is such that J is not well defined in all the space $W_0^{1,p}(\Omega)$. This kind of non-differentiable functional J that combines a critical point theory has been investigated in [8] for $p = 2$. In fact, the function J is well defined in $W_0^{1,p}(\Omega) \cap Ł^{\infty}(\Omega)$, if we impose an additional condition on γ, namely, $\gamma + p < p^*$, Furthermore,one has studied the similar quasilinear Euler-Lagrange equation in [1] for $p = 2$. Our technique for solving a quasilinear Euler-Lagrange equation (1.1)-(1.2) is based on approximating J with the sequence of functionals $J_{m,n}$ whose quadratic part in ∇v is bounded with respect to v. We point out that our approach has been studied in [1], including $L^{\infty}(\Omega)$ a priori estimates, allows to prove when $\gamma > 1$ that a critical point $u_{\overline{m},\overline{n}}$ of $J_{\overline{m},\overline{n}}$ for $\overline{m}, \overline{n}$ large enough, is a solution of (1.1)-(1.2) without passing to the limit on m and n. Hence we use the theorem 2.8 in [3] to prove the existence of infinitely many solutions of equation (1.1)-(1.2) for $0 < \lambda < \widetilde{\lambda}_0$ and $1 < \theta < p < q < \frac{p*}{p}(\gamma + p)$. we notice that the multiplicity results for p-Laplacian with critical growth of concave-convex functions has been intensively studied. Recently, the existence of the nonnegative bounded weak solution for the quasilinear Euler-Lagrange equation involving concave-convex nonlinearities with $p = 2$ has been investigated by David Arcoya and Lucio Boccardo (see [1,8]).Finally, the novelty of this paper is that we study the existence of Multiplicity bounded weak solutions for quasilinear Euler-Lagrange equation with $1 < p < N$.

Notation. in the rest of this work we make use of the following notation. $L^p(\Omega)$, $1 \leq p \leq \infty$, denote the Lebesgue spaces. The usual norm in $L^p(\Omega)$ is denoted by $| \ \ |_p$.

$W_0^{k,p}(\Omega)$ denote the Sobolev spaces. The norm in $W_0^{1,p}(\Omega)$ is denoted by $\| \ \|_p$.

$C_0, C_1, C_2, C_3, ...$ denote (possibly different) positive constants.

2. Main Result

Definition 2.1. A measurable function u is called a weak solution to the equation (1.1)-(1.2), if $u \in W_0^{1,p}(\Omega)$ such that $\mid u \mid^{\gamma-2} u \mid \nabla u \mid^p \in L^1(\Omega)$ and

$$\int_\Omega (a(x)+ \mid u \mid^\gamma) \mid \nabla u \mid^{p-2} \nabla u \nabla v + \frac{\gamma}{p} \int_\Omega \mid u \mid^{\gamma-2} u \mid \nabla u \mid^p v$$
$$= \lambda \int_\Omega \mid u \mid^{\theta-2} uv + \int_\Omega \mid u \mid^{q-2} uv$$

holds for every $v \in W_0^{1,p}(\Omega) \cap Ł^\infty(\Omega)$.

The main result of this paper is based on the existence of multiplicity bounded weak solutions to the equation (1.1)-(1.2) for that the main result is given by the following theorem.

Theorem 2.1. *Suppose that γ satisfies the condition that $\gamma + p < p^*$. Moreover, there exists $\widetilde{\lambda}_0$ such that*

$$1 < \theta < p < q < \frac{p*}{p}(\gamma + p); \quad 0 < \lambda < \widetilde{\lambda}_0. \tag{2.2}$$

Then, there exist infinitely many weak solutions for the problem (1.1)-(1.2).

Proof of Theorem 2.1. We use the theorem 2.8 in [3] in order to prove the existence of multiplicity weak solutions to the problem (1.1)-(1.2). Then we will divide the proof into several steps.

Step 1. A truncated function.

If m is positive integer, we consider the truncated function at level m, $T_m(t)$ is given

$$T_m(t) = \begin{cases} -m - \frac{1}{2}, & \text{if } \ t \leq -m-1, \\ (m+1)t + \frac{t^2+m^2}{2}, & \text{if } \ -m-1 \leq t \leq -m \\ t, & \text{if } \ -m \leq t \leq m, \\ (m+1)t - \frac{t^2+m^2}{2}, & \text{if } \ m \leq t \leq m+1, \\ m + \frac{1}{2}, & \text{if } \ t \geq m+1. \end{cases} \tag{2.3}$$

The function with same definition is is introduced in [1].

Assuming that q_0 and q_1 are two numbers such that $1 < q_0 < \theta < 2 < q_1 < q$ and the truncated function $f_{n,\lambda}(t)$ is defined by

$$f_{n,\lambda}(t) = \lambda h_n(t) + g_n(t),$$

where

$$h_n(t) = \begin{cases} \frac{|t|^\theta}{\theta}, & \text{if} \quad | t |< n, \\ n^\theta\Big(\frac{1}{\theta} - \frac{1}{q_0}\Big) + n^{\theta - q_0}\frac{|t|^{q_0}}{q_0}, & \text{if} \quad | t |\geq n, \end{cases} \tag{2.4}$$

$$g_n(t) = \begin{cases} \frac{|t|^q}{q}, & \text{if} \quad | t |< n, \\ n^q\Big(\frac{1}{q} - \frac{1}{q_1}\Big) + n^{q - q_1}\frac{|t|^{q_1}}{q_1}, & \text{if} \quad | t |\geq n. \end{cases} \tag{2.5}$$

By the definition of $h_n(t)$ and $g_n(t)$, we deduce the following inequalities

$$0 \leq h_n(t) \leq \frac{n^{\theta - q_0}}{q_0} | t |^{q_0} \quad \text{and} \quad 0 \leq h_n(t) \leq \frac{| t |^\theta}{\theta}, \tag{2.6}$$

$$0 \leq g_n(t) \leq \frac{n^{q - q_1}}{q_1} | t |^{q_1} \quad \text{and} \quad 0 \leq g_n(t) \leq \frac{| t |^q}{q}. \tag{2.7}$$

Consequently, we are able to deduce the estimate of $f_{n,\lambda}(t)$ by

$$0 \leq f_{n,\lambda}(t) \leq \frac{\lambda n^{\theta - q_0}}{q_0} \mid t \mid^{q_0} + \frac{n^{q - q_1}}{q_1} \mid t \mid^{q_1}, \quad \forall \mid t \mid\geq 0. \tag{2.8}$$

Let us consider the truncated functional

$$J_{m,n}(u) = \frac{1}{p}\int_\Omega (a(x) + \mid T_m(u) \mid^\gamma) \mid \nabla u \mid^p - \int_\Omega f_{n,\lambda}(u) \quad for \quad u \in W_0^{1,p}(\Omega), \tag{2.9}$$

which is clearly well defined since $q_0 < q_1 < p^*$.

Step 2. Geometry of truncated function.

Let r a positive real constant such that

$$B_r = \{u \in W_0^{1,p}(\Omega) / \parallel u \parallel_p \leq r\}.$$

The fact that, $a(x) + \mid T_m(u) \mid^\gamma \geq \alpha$ and combining with (2.8), we receive

$$\int_\Omega f_{n,\lambda}(u) \leq \lambda C_0 n^{\theta - q_0} \parallel u \parallel_p^{q_0} + C_1 n^{q - q_1} \parallel u \parallel_p^{q_1}. \tag{2.10}$$

Here C_0 and C_1 are nonnegative constants.

Performing the calculations and taking into account inequality (2.10), we can deduce by using the sobolev and Hölder inequalities the result below

$$J_{m,n}(u) \geq \frac{\alpha}{p} \parallel u \parallel_p^p - \lambda C_0 n^{\theta - q_0} \parallel u \parallel_p^{q_0} - C_1 n^{q-q_1} \parallel u \parallel_p^{q_1},$$

with $a(x) + \mid T_m(u) \mid^{\gamma} \geq \alpha$.

Therefore, there exist nonnegative constants $r_{n,\lambda}$, $\overline{r}_{n,\lambda}$ and $\widetilde{\lambda}_0$ such that:

$$J_{m,n}(u) > 0 \text{ in } B_{r_{n,\lambda}} \quad \text{and} \quad J_{m,n}(u) \geq \overline{r}_{n,\lambda} \text{ in } \partial B_{r_{n,\lambda}}.$$

for all $0 < \lambda < \widetilde{\lambda}_0$.

Step 3. Compactness of the truncated function.

Let $\{w_k\}$ be a sequence in $W_0^{1,p}(\Omega) \cap Ł^{\infty}(\Omega)$ satisfying, for every $n \in N$ the following conditions:

$$J_{m,n}(w_k) \leq C_1,$$

$$\mid w_k \mid_{\infty} \leq 2b_k,$$

$$\langle J'_{m,n}(w_k), w \rangle \leq \varepsilon_k \left(\frac{\mid w \mid_{\infty}}{b_k} + \parallel w \parallel_p \right),$$

for all $w \in W_0^{1,p}(\Omega) \cap Ł^{\infty}(\Omega)$. Here C_1 is a nonnegative constant, $\{b_k\} \subset R^+ - \{0\}$ is a nonnegative sequence and $\{\varepsilon_k\} \subset R^+ - \{0\}$ is a sequence converging to zero, then $\{w_k\}$ has a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Indeed, let

$$\widetilde{g}(\lambda, t) = \frac{\frac{1}{p+\gamma} + f_{n,\lambda}(t)}{t f'_{n,\lambda}(t)}$$

and

$$g_0(\lambda) = \max_{t \in R} \widetilde{g}(\lambda, t),$$

where $\lambda > 0$ and $t > 0$.

Let $\varepsilon > 0$ be given and choose $t_0 > 0$ so that

$$\max_{t \in R} \widetilde{g}(\lambda, t) \leq \widetilde{g}(\lambda, t_0) + \varepsilon$$

Clearly $\widetilde{g}(\lambda, t_0)$ is an increasing and continuous function with respect λ so that there exists $\widetilde{\lambda}_0$ a nonnegative number such that $0 < \widetilde{\lambda}_0 < \infty$ and such that

$$\widetilde{g}(\widetilde{\lambda}_0, t_0) \leq \frac{1}{p+\gamma} \quad \text{for all} \quad 0 < \lambda < \widetilde{\lambda}_0.$$

Therefore
$$\max_{t\in R}\widetilde{g}(\lambda,t)\leq\widetilde{g}(\lambda,t_0)+\varepsilon\leq\widetilde{g}(\widetilde{\lambda}_0,t_0)+\varepsilon.$$

Thus
$$\max_{t\in R}\widetilde{g}(\lambda,t)\leq\frac{1}{p+\gamma}+\varepsilon\quad\text{for all}\quad 0<\lambda<\widetilde{\lambda}_0.$$

Then it is easily checked by induction that
$$\frac{f_{n,\lambda}(t)}{tf'_{n,\lambda}(t)}<g_0(\lambda)<\frac{1}{p+\gamma}\quad for\quad all\quad 0<\lambda<\widetilde{\lambda}_0$$

After performing calculations of the term
$$J_{m,n}(w_k)-g_0(\lambda)\langle J'_{m,n}(w_k),w_k\rangle,$$

we obtain
$$\begin{aligned}\left(\frac{1}{p}-g_0(\lambda)\right)&\int_\Omega a(x)\mid\nabla w_k\mid^p\\&+\int_\Omega\left(\frac{1}{p}-g_0(\lambda)-\frac{\gamma}{p}g_0(\lambda)w_k\frac{T'_m(w_k)}{T_m(w_k)}\right)\mid T_{m)}(w_k)\mid^\gamma\mid\nabla w_k\mid^p\\&+\int_\Omega\big(g_0(\lambda)w_kf'_{n,\lambda}(w_k)-f_{n,\lambda}(w_k)\big)\leq C_1+\varepsilon_k\left(\frac{\mid w_k\mid_\infty}{b_k}+\parallel w_k\parallel_p\right).\end{aligned}$$

Notice that all the left-hand side terms are positives.In fact,the first one is nonnegative due to consequence of the definition of $g_0(\lambda)$, namely,$\frac{f_{n,\lambda}(t)}{tf'_{n,\lambda}(t)}<g_0(\lambda)<\frac{1}{p+\gamma}$. For the second term, it is enough to use $0\leq\frac{tT'_m(t)}{T_m(t)}\leq 1$ and that $g_0(\lambda)<\frac{1}{p+\gamma}$. The positiveness of the third term is easily verified by using the definition of $g_0(\lambda)$. Thereby, we are able to conclude that the sequence $\{w_k\}$ is bounded in $W_0^{1,p}(\Omega)$ for every p such that $1<p<N$. Furthermore, it weakly converges into $W_0^{1,p}(\Omega)$ up to the subsequence that we still denote $\{w_k\}$ converging to a function w.

Step 4. Existence of critical points of the truncated function.

We point out that the main idea of this proof is in [4], therefore we adapt the arguments of Theorem 2.8 in [3] to prove the existence of multiplicity critical points of $J_{m,n}$.

Let H_k be a k-dimensional subspace of $W_0^{1,p}(\Omega)$ as we take $w_k\in H_k$, the norm of w_k, $\parallel w_k\parallel_p$ is finite.

We set

$$\Sigma = \{C \subset W_0^{1,p}(\Omega) / \ \ 0 \overline{\in} C \ \ C = -C\},$$

for $C \in \Sigma$ the Z_2-genus of C is denoted by $\gamma(C)$.

According to the step 2 and step 3, the assumptions (I_1) and (I_3) of Theorem 2.3 (see [3])hold true.

Moreover, letting

$$A_{m,n} = B_{r_{n,\lambda}} \cup \{J_{m,n} \geq 0\}.$$

We can clearly assert that $H_k \cap A_{m,n}$ is bounded for all $n \in N$, the assumption (I_5) Theorem 2.3 is complete.

Next we set

$$\Gamma^* = \{h \in C(W_0^{1,p}(\Omega), W_0^{1,p}(\Omega)) : \\ h \text{ is an odd homeomorphism } h(0) = 0 \text{ and } h(B_1) \subset A_{m,n}\}$$

and

$$\Gamma_k = \{K \in \Sigma : \gamma(K \cap h(\partial B_1)) \geq k \ \ \forall h \in \Gamma^*\}.$$

and then

$$S_k = \inf_{K \in \Gamma_k} \max_{w \in K} J_{m,n}(w).$$

After that we can state that Lemma 2.7 in [3] still holds. Then choosing now

$$h(w) = r_{n,\lambda} w,$$

where $r_{n,\lambda}$ the nonnegative real is found in the step 2. h belongs to Γ^*, therefore, we could deduce that $K \cap B_{r_{n,\lambda}} \neq \emptyset$ for all $K \in \Gamma_k$.

Since $J_{m,n}$ is bounded from below on $\partial B_{r_{n,\lambda}}$, consequently

$$S_k = \inf_{K \in \Gamma_k} \max_{w \in K} J_{m,n}(w) \geq a_{n,\lambda} > 0.$$

Finally, all of assumptions of Theorem 2.8 in [3] are satisfied, so that there exist infinitely many critical points of $J_{m,n}$, Hence, the Dirichlet problem (2.11)-(1.2) possesses infinitely many nontrivial weak solutions.

Step 5. Uniformly L^∞ estimates.

We consider the following equation:

$$- div\{(a(x)+ \mid T_m(w_{m,n}) \mid^{\gamma}) \mid \nabla w_{m,n} \mid^{p-2} \nabla w_{m,n}\} \\ + \frac{\gamma}{p} \frac{T'_m(w_{m,n})}{T_m(w_{m,n})} \mid T_m(w_{m,n}) \mid^{\gamma} \mid \nabla w_{m,n} \mid^p = f'_{n,\lambda}(w_{m,n}). \quad (2.11)$$

Assume that either $w_{m,n} = u_{m,n}$, or $w_{m,n} = u^0_{m,n}$, ... $w_{m,n} = u^k_{m,n}$, or ... are solution of (2.11)-(1.2). Setting $T_m(w_{m,n}) = w_{m,n}$ and $v =| w_{m,n} |^b w_{m,n}$ as a test function, we have

$$(b{+}1)\int_\Omega (a(x){+} \mid w_{m,n} \mid^\gamma) \mid \nabla w_{m,n} \mid^p \mid w_{m,n} \mid^b {+} \frac{\gamma}{p}\int_\Omega \mid w_{m,n} \mid b+\gamma \mid \nabla w_{m,n} \mid^p \leq (\lambda+1)n^{\theta-q_0+q-q_1}\int_\Omega \mid w_{m,n} \mid^{b+q}. \quad (2.12)$$

Dropping the positive terms on the left hand side of (2.12), we get

$$(b+1)\int_\Omega a(x) \mid w_{m,n} \mid^b \mid \nabla w_{m,n} \mid^p \leq (\lambda+1)n^{\theta-q_0+q-q_1}\int_\Omega \mid w_{m,n} \mid^{b+q}. \quad (2.13)$$

On the other hand, we obtain the following result by using the sobolev inequality

$$C_2^p\left(\int_\Omega \mid w_{m,n} \mid^{\frac{b+p}{p}p^*}\right)^{\frac{p}{p^*}} \leq \frac{(b+p)^p}{p^p}\int_\Omega \mid w_{m,n} \mid^b \mid \nabla w_{m,n} \mid^p,$$
$$C_2^p\left(\int_\Omega \mid w_{m,n} \mid^{\frac{b+p}{p}p^*}\right)^{\frac{p}{p^*}} \leq \frac{(b+p)^p}{p^p}\frac{1}{\alpha}\int_\Omega a(x) \mid w_{m,n} \mid^b \mid \nabla w_{m,n} \mid^p. \quad (2.14)$$

Combining (2.13) with (2.14), we receive

$$\left(\int_\Omega \mid w_{m,n} \mid^{\frac{b+p}{p}p^*}\right)^{\frac{p}{p^*}} \leq \left(\frac{(b+p)^p(\lambda+1)}{\alpha(pC_2)^p(b+1)}\right) n^{\theta-q_0+q-q_1}\int_\Omega \mid w_{m,n} \mid^{b+q}.$$

Therefore

$$\mid w_{m,n} \mid^{b+q}_{\frac{b+p}{p}p^*} \leq \left(\frac{(b+p)^p(\lambda+1)}{\alpha(pC_2)^p(b+1)}\right) n^{\theta-q_0+q-q_1} \mid w_{m,n} \mid^{b+q}_{b+q}.$$

Let $r = b+q$, then

$$\mid w_{m,n} \mid_{\frac{r-q+p}{p}p^*} \leq \left(\frac{(r-q+p)^p(\lambda+1)}{\alpha(pC_2)^p(r-q+1)}\right)^{\frac{1}{r-q+p}} n^{\frac{\theta-q_0+q-q_1}{r-q+p}} \mid w_{m,n} \mid_r^{\frac{r}{r-q+p}}.$$

Notice that $w_{m,n}$ belongs to $W_0^{1,p}(\Omega)$ and so to $L^{p^*}(\Omega)$, we can choose $r = r_0 = p^* - q$ to deduce that $w_{m,n}$ belongs to $L^{\frac{r_0+p-q}{p}p^*}(\Omega)$, we can then choose $r = r_1 = \frac{r_0-q+p}{p}p^*$ to obtain $w_{m,n}$ belongs to $L^{\frac{r_0+p-q}{p}p^*}(\Omega)$. Iterating this process and defining by induction r_k as

$$\begin{cases} r_0 = p^* - q, \\ r_k = r_{k-1}\frac{p^*}{p} + \frac{p^*}{p}(p-q). \end{cases} \quad (2.15)$$

We infer that $w_{m,n}$ belongs to $L^{r_k}(\Omega)$ with

$$| w_{m,n} |_{r_k} \leq \left(\frac{(r_k - q + p)^p(\lambda+1)}{\alpha(pC_2)^p(r_k - q + 1)}\right)^{\frac{1}{r_k-q+p}} n^{\frac{\theta - q_0 + q - q_1}{r_k - q + p}} \mid w_{m,n} \mid^{\frac{p^*}{p}\frac{r_{k-1}}{r_k}} .$$

Therefore

$$| w_{m,n} |_{r_k} \leq \ldots\ldots \leq C_3 \mid w_{m,n} \mid_{p^*}^{(\frac{p^*}{p})^k \frac{p^*-q}{r_k^k}} \leq C_4,$$

because

$$\int_\Omega a(x) \mid \nabla w_{m,n} \mid^p \quad \text{is bounded with respect to } m \text{ and } n.$$

Since $\frac{p^*}{p} > 1$, it is enough to show that r_k is increasing sequence which diverges to infinity, thus, if it is such that $\frac{r_k+p-q}{p}p^* \geq \frac{N}{2}$ an adaptation to the quasilinear case of the proof of a result of Stampacchia (see [5]) implies that there exists $M_n > 0$ such that

$$| w_{m,n} |_\infty \leq M_n.$$

Let m_n be an integer such that $m_n \geq max(M_n + p, \bar{t})$, if we define $w_n \stackrel{def}{=} w_{\overline{m},n}$, namely, either $w_n = u_n^0 \stackrel{def}{=} u^0_{\overline{m},n}$ or $w_n = u_n^1 \stackrel{def}{=} u^1_{\overline{m},n}$ or $\ldots\ldots$ or $w_n = u_n^k \stackrel{def}{=} u^k_{\overline{m},n}$ or $w_n = \ldots \stackrel{def}{=} \ldots$. Then $T_{m_n}(w_n) = w_n$ and $T'_{m_n}(w_n) = 1$, consequently, the equation which is satisfied by w_n is

$$-div\big((a(x)+ \mid w_n \mid^\gamma) \mid \nabla w_n \mid^{p-2} \nabla w_n \mid\big) + \frac{\gamma}{p} \mid w_n \mid^{\gamma-2} w_n \mid \nabla w_n \mid^p = f'_{n,\lambda}(w_n), \quad (2.16)$$

with zero Dirichlet boundary condition.

Notice that by the assumption $q < \frac{p^*}{p}(p+\gamma)$, then w_n is bounded in $L^q(\Omega)$ using this fact, we are going to show that w_n is uniformly bounded in $L^\infty(\Omega)$.

Let $b > 0$ as before, and choose $v =\mid w_n \mid^b w_n$ as a test function in the equation (2.16)-(1.2) satisfied w_n.

The fact that $f'_{n,\lambda}(t) \leq (\lambda+1) \mid t \mid^{q+b-1}$ and we drop two nonnegative terms then we obtain

$$(b+1)\int_\Omega a(x) \mid w_n \mid^{b+\gamma} \mid \nabla w_n \mid^p \leq (\lambda+1)\int_\Omega \mid w_n \mid^{q+b}.$$

However, we obtain another inequality when we apply the sobolev inequality to $w_n^{b+p+\gamma}$

$$C_5^p\left(\int_\Omega \mid w_n \mid^{\frac{\gamma+b+p}{p}p^*}\right)^{\frac{p}{p^*}} \leq \left(\frac{\gamma+b+p}{p}\right)^p \int_\Omega \mid w_n \mid^{b+\gamma} \mid \nabla w_n \mid^p .$$

Therefore

$$\left(\int_\Omega \mid w_n \mid^{\frac{\gamma+b+p}{p}p^*}\right)^{\frac{p}{p^*}} \leq \frac{(\gamma+b+p)^p(\lambda+1)}{(pC_5)^p(b+1)} \int_\Omega \mid w_n \mid^{b+q}.$$

Here w_n belongs to $L^{\frac{\gamma+r-q+p}{p}p^*}(\Omega)$ provided that w_n belongs to $L^r(\Omega)$ with $b = r - q$ then

$$\mid w_n \mid_{\frac{\gamma+b+p}{p}p^*} \leq \left(\frac{(\gamma+b+p)^p(\lambda+1)}{(pC_5)^p(b+1)}\right)^{\frac{p^*}{p}} \mid w_n \mid_r^{\frac{r}{\gamma+r-q+p}},$$

because $\frac{p}{b+\gamma+p} \leq p^*$.

Arguing as before, if we consider the sequence r_k as follows

$$\begin{aligned} r_0 =& \frac{p^*}{p}(\gamma+p), \\ r_k =& r_{k-1}\frac{p^*}{p} + \frac{p^*}{p}(\gamma+p-q). \end{aligned} \tag{2.17}$$

Thus $w_{m,n}$ belongs to $L^{r_k}(\Omega)$ for every k, and

$$\mid w_n \mid_{r_k} \leq \left(\frac{(\gamma+r_{k-1}-q+p)^p(\lambda+1)}{(pC_5)^p(r_{k-1}-q+1)}\right)^{\frac{p^*}{p}} \mid w_n \mid_{r_k}^{\frac{p^*}{p}\frac{r_{k-1}}{r_k}} .$$

Therefore

$$\mid w_n \mid_{r_k} \leq \ldots\ldots \leq C_6 \mid w_n \mid_{\frac{p^*}{p}(\gamma+p)}^{(\frac{p^*}{p})^{k+1}\frac{\gamma+p}{r_k^k}} \leq C_7.$$

The fact that

$$\int_\Omega \mid w_n \mid^\gamma \mid \nabla w_n \mid^p \quad \text{is bounded with respect to } n.$$

Namely $w_k \in L^{\frac{p^*}{p}(\gamma+p)}(\Omega)$, clearly the sequence $\{r_k\}$ is increasing and unbounded since $\frac{p^*}{p} > 1$. So that in a finite number of steps we conclude that

$\lambda \mid w_n \mid^{\theta-2} w_n + \mid w_n \mid^{q-2} w_n$ is bounded in L^r with $r > \frac{N}{2}$. Using again an adaptation of the proof theorem 2.1 in [5] yields that there exists a nonnegative constant $C_0' > 0$ such that

$$\mid w_n \mid_\infty \leq C_0', \quad \forall n \geq \max(\bar{t}, \overline{n}).$$

In other words, we obtain

$$\mid u_n^0 \mid_\infty \leq C_1', \quad \mid u_n^1 \mid_\infty \leq C_2', \ldots, \mid u_n^k \mid_\infty \leq C_k' \ldots, \quad \forall n \geq \max(\bar{t}, \overline{n}).$$

Step 6. Conclusion

Finally, $\forall n \geq \max(C_0', \bar{t}, \overline{n})$, we have

$$f'_{n,\lambda}(w_n) = \lambda \mid w_n \mid^{\theta-2} w_n + \mid w_n \mid^{q-2} w_n$$

and

$$w \stackrel{def}{=} w_{\overline{n}}.$$

In other words, either

$$w \stackrel{def}{=} u^0 \stackrel{def}{=} u^0_{\overline{n}}, \quad \text{or} \quad w \stackrel{def}{=} u^1 \stackrel{def}{=} u^1_{\overline{n}}, \quad \text{or} \ldots \text{or} \quad w \stackrel{def}{=} u^k \stackrel{def}{=} u^k_{\overline{n}}, \ldots .$$

Hence, we can conclude that the problem (1.1)-(1.2) has an infinitely many positive bounded weak solutions. □

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