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REMARKS ON THE KLEIN-GORDON EQUATION IN A ROBERTSON-WALKER UNIVERSE

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Abstract: We present a class of solutions of certain radial equations associated to Klein-Gordon equations in a closed Robertson-Walker universe.

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1. Introduction

The standard model in cosmology is the homogeneous, isotropic space-time Friedmann model $Y = X \times (\alpha, \beta)$ with a Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 d\phi^2) \right] \tag{1.1}$$

for a suitable choice of coordinates (t, ϕ, θ, r) on Y, where c is the speed of light, $a(t)$ is a scale factor defined on the open interval (α, β) of the real line $\mathbb{R}$, and $k = 0, 1$, or -1 corresponding to whether the constant curvature of the spatial section X of Y is zero, positive, or negative,

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respectively. Thus we may take $X = R^3, S^3$ (the 3-sphere of unit radius), or hyperbolic 3-space, respectively. This metric describes very well the expanding universe. Early stages of the universe, for example, are described by taking $k = 0$ and $a(t) \sim t^{1/2}$, [2].

Given a pseudo-Riemannian metric on any manifold, there is a corresponding Laplace-Beltrami operator $\Box$. In particular for the space-time metric (1.1) on Y, $\Box$ is given by

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{3}{c^2}\frac{\dot{a}(t)}{a(t)}\frac{\partial}{\partial t} + \frac{\triangle}{a^2(t)}\,, \tag{1.2}$$

where $\triangle$ is the Laplace-Beltrami operator on X, given by (1.13) below for the case that will be of interest to us, namely the case $X = S^3$. $\dot{a}(t)$ denotes differentiation of $a(t)$ with respect to time t. The Klein-Gordon equation with mass m assumes the form

$$(\Box - m^2)\psi = 0 \quad \text{on} \quad Y. \tag{1.3}$$

If one separates the space and time variables as usual, writing

$$\begin{aligned} \psi(x,t) = F(t)\phi(x)\,, \qquad & t \in (\alpha,\beta), x \in X, \\ & F \in C^\infty(\alpha,\beta), \phi \in C^\infty(X), \end{aligned} \tag{1.4}$$

one finds that, by (1.2), (1.3) is equivalent to the system

$$-\frac{1}{c^2}\ddot{F}(t) - \frac{3}{c^2}\frac{\dot{a}(t)}{a(t)}\dot{F}(t) + \left[\frac{\nu}{a^2(t)} - m^2\right]F(t) = 0, \tag{1.5}$$

$$\triangle\phi = \nu\phi\,, \tag{1.6}$$

for some scalar ν. The first derivative $\dot{F}(t)$ of $F(t)$ appears in (1.5). Note however that if we set

$$H(t) = a(t)^{3/2}F(t) \quad \text{on} \quad (\alpha,\beta)\,, \tag{1.7}$$

we obtain the following alternate version of (1.5), where the first derivative $\dot{H}(t)$ of $H(t)$ does not appear:

$$\ddot{H}(t) + \left[-\frac{3}{4}\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{3}{2}\frac{\ddot{a}(t)}{a(t)} - \frac{\nu c^2}{a^2(t)} + m^2c^2\right]H(t) = 0. \tag{1.8}$$

We shall confine our attention to a closed universe, where $k = 1$. The discussion for an open universe with $k = -1$ proceeds along similar lines. For $k = 1$, $\triangle$ is the Laplace-Beltrami operator on $X = S^3$ with its usual metric induced from that of R^4; see formula (1.13). By a further separation of variables one can express $\phi(x)$ as

$$\phi(x) = R(r)Z(\phi, \theta), \tag{1.9}$$

where $Z(\phi, \theta)$ is a suitable eigenfunction (a spherical harmonic), and where the radial function $R(r)$ satisfies the differential equation

$$(1 - r^2)R''(r) + \left(\frac{2}{r} - 3r\right) R'(r) - \frac{\ell(\ell + l)}{r^2} R(r) = \nu R(r) \tag{1.10}$$

on (0,1) for $\ell \geq 0$ an integer; see [1], [5]. More precisely, equation (1.9) means that if we write for $x = (x_1, x_2, x_3, x_4) \in S^3$

$$\begin{aligned} &x_1 = r \sin\theta \sin\phi\ ,\ x_2 = r \sin\theta \cos\phi\ ,\ x_3 = r\cos\theta \\ &x_4 = \sqrt{1 - r^2},\ \text{ with }\ 0 < \phi < 2\pi,\ \ 0 < \theta < \pi,\ \ 0 < r < 1, \end{aligned} \tag{1.11}$$

then

$$\phi(x) = R(r)Z(\phi, \theta) = R(r)P_\ell(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)\,, \tag{1.12}$$

where P_ℓ is the restriction to S^2 of a harmonic, homogeneous polynomial on R^3 of degree ℓ. This is the ℓ which appears in (1.10), where the factor $-\ell(\ell + 1)$ is an eigenvalue of the Laplace-Beltrami operator $\triangle_2$ on S^2: $\triangle_2 P_\ell = -\ell(\ell + 1)P_\ell$. (ϕ, θ, r) are the coordinates used in (1.1).

In the present paper our remarks deal with a method of obtaining an infinite family of solutions $\{R_n\}_{n=0}^\infty$ of the radial equation (1.10), for a corresponding sequence $\{\nu_n\}_{n=0}^\infty$ of values ν. This equation appears in the work [1], but is not solved there. We shall construct, in fact, solutions of a more general equation (see equation (2.1)) which have applications indicated in [5]. By various changes of variables we shall reduce equation (1.10) (or more generally equation (2.1)) to an equation of hypergeometric type, in a generalized sense. We then apply the theory developed in [3] to obtain the desired solutions, under an elegant integrality condition which also yields an explicit form of the eigenvalues ν. These solutions tensored with the spherical harmonics P_ℓ in (1.12) and the $F(t) = a(t)^{-3/2}H(t)$ (for solutions $H(t)$ of (1.8)) provide for solutions $\psi(x,t)$ in the form (1.4) of the Klein-Gordon equation (1.3). Explicit solutions of (1.8) however are not generally available. One usually resorts to semi-classical approximation methods. For the sake of completeness we note that for the choice of coordinates (ϕ, θ, r) in (1.11), Δ is given by the formula

$$\Delta = (1-r^2)\frac{\partial^2}{\partial r^2} + \left(\frac{2}{r} - 3r\right)\frac{\partial}{\partial r} + \frac{\Delta_2}{r} \tag{1.13}$$

with Δ_2 (as above) given by

$$\Delta_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{1.14}$$

2. Statement and Proof of the Main Result

Consider the following differential equation on (0,1):

$$(1-x^2)R''(x) + \left(\frac{\alpha-1}{x} - \alpha x\right)R'(x) - \frac{\ell(\ell+\alpha-2)R(x)}{x^2} = \gamma R(x)\,, \tag{2.1}$$

where $\alpha > 2$, $\ell \geq 0$; ℓ need not be an integer. For $\alpha = 3$ and $\gamma = \nu$, equation (2.1) reduces to equation (1.10). For $r > 0$, $n = 0, 1, 2, 3, ...$ define

$$\Phi_n(r) = (1+r^2)^{\frac{n+\ell+\alpha-2}{2}}\frac{d^n}{dr^n}(1+r^2)^{-(\ell+\frac{\alpha-1}{2})}, \tag{2.2}$$

$$\tilde{\lambda}_n = (n+\ell)(n+\ell+\alpha-1). \tag{2.3}$$

Then we have the following main result.

Theorem 2.1. *Assume that for γ in (2.1) and $\tilde{\lambda}_n$ in (2.3) the following integrality condition is satisfied:*

$$-\gamma = \tilde{\lambda}_n. \tag{2.4}$$

Define R_n on (0,1) by

$$R_n(x) = \frac{1}{x}\Phi_n\left(\frac{\sqrt{1-x^2}}{x}\right) \tag{2.5}$$

for Φ_n in (2.2). Then R_n is a solution of equation (2.1).

Proof. For $y(x) \overset{def}{=} xR(x)$ on (0,1), equation (2.1) implies the following differential equation for y:

$$\begin{aligned}(1-x^2)y''(x) &+ \left[\frac{\alpha-3}{x} + (2-\alpha)x\right]y'(x) \\ &+ \left(\frac{\beta-\alpha+3}{x^2} + \alpha - 2 - \gamma\right)y(x) = 0,\end{aligned} \tag{2.6}$$

where $\beta \overset{def}{=} -\ell(\ell+\alpha-2)$. Next define $v(t)$ on $(0, \pi/2)$ by $v(t) \overset{def}{=} y(\sin t)$. Given equation (2.6) one has that $v(t)$ satisfies

$$v''(t) + (\alpha-3)(\cot t)v'(t) + \left(\frac{C}{\sin^2 t} + A\right)v(t) = 0 \tag{2.7}$$

for $A \overset{def}{=} \alpha{-}2{-}\gamma$, $C \overset{def}{=} \beta{-}\alpha{+}3$. Finally for $\Phi(r) \overset{def}{=} v(\cot^{-1} r), r > 0$, equation (2.7) implies that $\Phi(r)$ satisfies

$$\sigma^2(r)\Phi''(r) + \sigma(r)\tilde{\tau}(r)\Phi'(r) + \tilde{\sigma}(r)\Phi(r) = 0\,, \tag{2.8}$$

where $\sigma(r) = 1 + r^2, \tilde{\tau}(r) = (5-\alpha)r, \tilde{\sigma}(r) = C(1+r^2) + A$.

An equation of the form (2.8), where $\sigma(r), \tilde{\sigma}(r), \tilde{\tau}(r)$ are polynomial functions of r with degree $\sigma(r), \tilde{\sigma}(r) \leq 2$ and degree $\tilde{\tau}(r) \leq 1$, is said to be of *hypergeometric type*, in a generalized sense. Thus by the above changes of variables we have transformed equation (2.1) to an equation of hypergeometric type, the point being that a nice theory exists, due to Nikiforov and Uvarov [3], for solving such an equation. Details of this theory, which we now apply, are presented also in Chapter 4 of [4].

One begins by choosing a polynomial square root $p(r)$ of the polynomial

$$\begin{aligned} f_k(r) &\overset{def}{=} \frac{(\tilde{\tau}-\sigma')^2}{4}(r) + k\sigma(r) - \tilde{\sigma}(r) \\ &= \left[\frac{(\alpha-3)^2}{4} + k - C\right]r^2 + k - A - C\end{aligned} \tag{2.9}$$

for $k \in R$. For this, assume that the discriminant

$$\triangle(k) = -4\left[\frac{(\alpha-3)^2}{4} + k - C\right] \cdot (k - A - C)$$

of $f_k(r)$ vanishes:

$$f_k(r) = \begin{bmatrix} -A - \dfrac{(\alpha-3)^2}{4} \text{ if } \dfrac{(\alpha-3)^2}{4} + k - C = 0 \\ \left[\dfrac{(\alpha-3)^2}{4} + A\right] r^2 \text{ if } k - A - C = 0 \end{bmatrix}. \tag{2.10}$$

Now $A \stackrel{def}{=} \alpha - 2 - \gamma$ with $\alpha > 2$ and with $-\gamma \leq 0$ in particular if γ satisfies (2.4) (since $n, \ell \geq 0$). That is, $A > 0$ and for $k - A - C = 0$ (i.e. for the choice of $k = A + C$) we can clearly take

$$p(r) = Br \stackrel{def}{=} \left[\frac{(\alpha-3)^2}{4} + A\right]^{1/2} r \tag{2.11}$$

with $B > 0$.

Next set

$$\begin{aligned} \pi^{\pm}(r) &\stackrel{def}{=} \frac{(\sigma' - \tilde{\tau})}{2}(r) \pm p(r) \\ &= \left[\frac{(\alpha-3)}{2} \pm B\right] r, \end{aligned} \tag{2.12}$$

$$\tau(r) \stackrel{def}{=} \tilde{\tau}(r) + 2\pi^{\pm}(r) = 2(1 \pm B)r. \tag{2.13}$$

As we had two choices for a square root p of f_k, we could also choose either the plus or minus sign in (2.12), (2.13). We shall choose the minus sign. We set

$$\mu \stackrel{def}{=} k + (\pi^-)' = A + C + \frac{\alpha - 3}{2} - B, \tag{2.14}$$

$$\begin{aligned} \lambda_n &\stackrel{def}{=} -n\tau' - \frac{n(n-1)}{2}\sigma'' \\ &= -n + 2nB - n^2 \end{aligned} \tag{2.15}$$

for $n = 0, 1, 2, 3, \ldots$ Finally set

$$\phi(r) \overset{def}{=} e^{\int \pi^-(r)/\sigma(r)dr} = (1+r^2)^{\frac{1}{2}\left(\frac{\alpha-3}{2}-B\right)}, \tag{2.16}$$

$$h(r) \overset{def}{=} \int \frac{\tilde{\tau}(r)}{\sigma(r)} dr = \frac{(5-\alpha)}{2}\log(1+r^2), \tag{2.17}$$

$$\rho(r) \overset{def}{=} \frac{\phi(r)^2}{\sigma(r)} e^{h(r)} = (1+r^2)^{-B}. \tag{2.18}$$

Then ρ satisfies

$$(\rho\sigma)' = \rho\tau \tag{2.19}$$

and for

$$\begin{aligned} y_n(r) &\overset{def}{=} \frac{1}{\rho(r)} \frac{d^n}{dr^n} \sigma^n(r)\rho(r) \\ &= (1+r^2)^B \frac{d^n}{dr^n}(1+r^2)^{n-B} \end{aligned} \tag{2.20}$$

it is shown in [3],[4] that $y_n(r)$ (a polynomial in r of degree $\leq n$) is a solution of

$$\sigma\omega'' + \tau\omega' + \mu\omega = 0 \tag{2.21}$$

provided

$$\mu = \lambda_n. \tag{2.22}$$

Moreover for Φ and ω related by $\Phi = \phi\omega$, one has that Φ solves (2.8) if and only if ω solves (2.21). It follows that under the condition (2.22), which we call an integrality condition, $\Phi_n(r) \overset{def}{=} \phi(r)y_n(r)$ is a solution of equation (2.8). Note that equation (2.21) is also an equation of hypergeometric type (since we may express it as $\sigma^2\omega'' + \sigma\tau\omega' + \mu\sigma\omega = 0$), which we call a canonical form of (2.8). To conclude the proof we therefore need to explicite the condition (2.22), and work backwards through the above variable changes to express R directly in terms of Φ. This will allow us to define a solution R_n of the original equation (2.1) (under the integrality condition).

By definitions (2.14), (2.15) condition (2.22) means that

$$A + C + \frac{\alpha - 3}{2} - B = -n + 2nB - n^2\,, \tag{2.23}$$

where $B^2 \overset{def}{=} \frac{(\alpha-3)^2}{4} + A$ by (2.11). That is, $A = B^2 - \frac{(\alpha-3)^2}{4}$ which by (2.23) leads to the quadratic equation

$$B^2 - (1+2n)B + \frac{(\alpha-3)(5-\alpha)}{4} + C + n(n+1) = 0 \tag{2.24}$$

for B with solutions

$$B = \frac{(1+2n) \pm [\alpha^2 - 8\alpha + 16 - 4C]^{1/2}}{2}. \tag{2.25}$$

On the other hand, $C \overset{def}{=} \beta - \alpha + 3 \overset{def}{=} -\ell(\ell+\alpha-2) - \alpha + 3 \Longrightarrow [\alpha^2 - 8\alpha + 16 - 4C]^{1/2} = ([2\ell + (\alpha-2)]^2)^{1/2} = 2\ell + \alpha - 2$ for $\ell \geq 0, \alpha \geq 2$, which by (2.25) gives

$$\begin{aligned} \frac{(\alpha-3)^2}{4} + \alpha - 2 - \gamma &\overset{def}{=} \frac{(\alpha-3)^2}{4} + A \\ \overset{def}{=} B^2 &= \left[\frac{(1+2n) \pm (2\ell + \alpha - 2)}{2}\right]^2. \end{aligned} \tag{2.26}$$

For the choice of the plus sign in (2.26) we get $-\gamma = (n+\ell)(n+\ell+\alpha-1)$. That is, condition (2.22) reduces exactly to condition (2.4). For the choice of the minus sign in (2.26) we get

$$-\gamma = (n-\ell)(n-\ell+3-\alpha) + (2-\alpha). \tag{2.27}$$

That is, there are actually two integrality conditions: (2.4) and (2.27). For physical reasons we shall focus first on solutions of equation (2.1) that derive from condition (2.4) - i.e. from the choice of the plus sign in (2.26). We mention, by the way, that in the special case when $\alpha = 3$, conditions (2.4) and (2.27) coincide with the quantization condition (i), (ii) in [1], which were found there by power series methods - methods of Frobenius.

Since $\Phi(r) \overset{def}{=} v(\cot^{-1} r)$ for $r > 0$, $v(t) = \Phi(\cot t)$ on $(0, \pi/2)$. Similarly $v(t) \overset{def}{=} y(\sin t) \Longrightarrow y(x) = v(arc\sin x) = \Phi(\cot\ arc\sin x) =$

$\Phi\left(\frac{\sqrt{1-x^2}}{x}\right)$ on (0,1), and $y(x) \stackrel{def}{=} xR(x) \Longrightarrow R(x) = y(x)/x = \frac{1}{x}\Phi\left(\frac{\sqrt{1-x^2}}{x}\right)$. Thus since Φ_n solves (2.8) under condition (2.22) if we set

$$R_n(x) \stackrel{def}{=} \frac{1}{x}\Phi_n\left(\frac{\sqrt{1-x^2}}{x}\right) \quad \text{on} \quad (0,1)\,, \tag{2.28}$$

we obtain a solution R_n of equation (2.1) for $-\gamma = (n+\ell)(n+\ell+\alpha-1)$, which as we have seen is condition (2.22). By definitions (2.16) and (2.20),

$$\begin{aligned}
&\Phi_n(r) \stackrel{def}{=} \phi(r)y_n(r)\\
&\stackrel{def}{=} (1+r^2)^{\frac{1}{2}\left(\frac{\alpha-3}{2}-B\right)}(1+r^2)^B\frac{d^n}{dr^n}(1+r^2)^{n-B}\\
&= (1+r^2)^{\frac{n+\ell+\alpha-2}{2}}\frac{d^n}{dr^n}(1+r^2)^{-\ell-\frac{(\alpha-1)}{2}}\,,
\end{aligned} \tag{2.29}$$

(as in (2.2)) since $B = n+\ell+\frac{\alpha-1}{2}$ by (2.26), where (recall) we have chosen the plus sign there. Thus the proof of Theorem 2.1 is concluded. ■

For the choice of the minus sign in (2.26) (i.e. in (2.25)), $B = \frac{3+2n-2\ell-\alpha}{2}$ and we have seen that condition (2.22) reduces to condition (2.27). Then by the second statement of equality in (2.29) we get for $r > 0$

$$\Phi_n^-(r) \stackrel{def}{=} \Phi_n(r) = (1+r^2)^{\frac{n-\ell}{2}}\frac{d^n}{dr^n}(1+r^2)^{\ell+\frac{\alpha-3}{2}}. \tag{2.30}$$

As a by-product of the proof of Theorem 2.1 we therefore obtain also the following theorem.

Theorem 2.2. *Assume that γ in (2.1) satisfies condition (2.27): $-\gamma = (n-\ell)(n-\ell+3-\alpha)+(2-\alpha)$. Define R_n^- on (0,1) by*

$$R_n^-(x) \stackrel{def}{=} \frac{1}{x}\Phi_n^-\left(\frac{\sqrt{1-x^2}}{x}\right) \tag{2.31}$$

for Φ_n^- defined in (2.30). Then R_n^- is a solution of equation (2.1). Here again we asume $\alpha > 2, \ell \geq 0$.

Since ν in (1.10) is an eigenvalue of $\triangle$ (by (1.6)) one knows that $-\nu$ must have the form $-\nu = p(p+2)$ for some integer $p \geq 0$. This is consistent with condition (2.4), where in (2.3) we take $\alpha = 3 : -\nu = \tilde{\lambda}_n \overset{def}{=} (n+\ell)(n+\ell+2) \Longrightarrow p = -1 \pm (n+\ell+1) \Longrightarrow$ we must have $p = n+\ell \geq \ell$ (since $p \geq 0$). On the other hand condition (2.27) requires (for $\alpha = 3$) $-\nu = (n-\ell)^2 - 1$, which implies that $p = -1 \pm |n-\ell|$. As $p \geq 0$, we must have $p = -1 + |n-\ell|$; i.e. $n \neq \ell$. For the choice of eigenvalues $\nu = -(n+\ell)(n+\ell+2), n = 0, 1, 2, 3, ...,$ for example (the choice made in [1] for physical reasons), one obtains solutions R_n equation (1.10) given by (2.5) of Theorem 2.1, where (again for $\alpha = 3$)

$$\Phi_n(r) \overset{def}{=} (1+r^2)^{\frac{n+\ell+1}{2}} \frac{d^n}{dr^n}(1+r^2)^{-(\ell+1)} \tag{2.32}$$

by (2.2).

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