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EXISTENCE PROBLEM OF PERIODIC SOLUTION OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEGREE THEORY

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Abstract:

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1. Introduction

In this paper, we study the following neutral functional differential equation with linear difference operator:

$$\frac{dD_2(x_t)}{dt} = f(t, x_t), \tag{1}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau = \max_{1\le k\le m}\{\tau_k\}$ is a constant,

$$D_2 : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n, D_2(\varphi) = \varphi(0) - B\sum_{k=1}^{m} \varphi(-\tau_k).$$

In this equation, $D_2 : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear, continuous, atomic zero, and $f \in C((\mathbb{R} \times C[-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$ with $f(t+\omega, \varphi) \equiv f(t, \varphi)$ for all $\varphi \in$

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$C([-\tau,0],\mathbb{R}^n)$ and for any bounded set $\Omega \subset C([-\tau,0],\mathbb{R}^n)$, $f([0,\omega]\times\Omega)$ is bounded in $\mathbb{R}^n$. From the theory of neutral functional differential equations, any solution $u(t)$ to Eq.(1) only implies that $D_2(u_t) = u(t) - B\sum_{k=1}^{m} u(t-\tau_k)$ is continuously differentiable in t, but generally $u(t)$ may not be differentiable in t. J. Hale gave an important definition named stable difference operator D [1, 2]; The linear difference operator $D : C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $D(\varphi) = \varphi(0) - \int_{-\tau}^{0} \varphi(\theta)d\mu(\theta)$ is called stable, if the zero solution of the difference equation $Dy_t = 0$, $y_0 = \varphi \in \{C([-\tau,0],\mathbb{R}^n) : D\varphi = 0\}$ is uniformly asymptotically stable. Under the condition that the linear difference operator D is stable, J. Hale obtained that any ω-periodic solution to the equation

$$\frac{dDx_t}{dt} = f(t,x_t)$$

has a continuous first derivatives [1].

2. Linear Difference Operators with Multiple Deviating Arguments

In this part, under the case neutral functional differential equations have multiple deviating arguments, we will study some properties of D_2-operator. Let $a = (a_1, a_2, ..., a_n)^T \in \mathbb{C}^n$ be a complex vector, $|a| = (\sum_{i=1}^{n} |a_i|^2)^{\frac{1}{2}}$, and $B = [b_{i,j}]_{n\times n}$, $|B| = (\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{i,j}|^2)^{\frac{1}{2}}$ be a complex matrix. Sets $C_\omega, C_\omega^1, P_\omega, P_\omega^1, T_\omega$ and T_ω^1 are defined as in [3]. Clearly, they are all Banach spaces. Let's consider, neutral functional differential equation

$$\frac{dD_2(x_t)}{dt} = f(t,x_t). \tag{2}$$

In this equations, D_2-operator defined by

$$D_2 : C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n \; ; \; D_2(\varphi) = \varphi(0) - B\Sigma_{k=1}^{m}\varphi(-\tau_k),$$

where $\tau = \max_{1\le k\le m}\{\tau_k\}$ and $\tau_k > 0$ is a constant for $k = 1,2,...,m$. So, from the definition of D_2-operator, for any $x_k \in C([-\tau,0],\mathbb{R}^n)$, we can write the equation

$$D_2(\varphi) = \varphi(0) - B\Sigma_{k=1}^{m}\varphi(-\tau_k),$$

that is

$$D_2(x_t) = x(t) - B\Sigma_{k=1}^{m}x(t-\tau_k).$$

If we define the operator

$$A : C_\omega \to C_\omega \; ; \quad [Ax](t) = x(t) - B\Sigma_{k=1}^{m}x(t-\tau_k), \tag{3}$$

it is easy to see that for any $h \in C_\omega$, the existence of continuous ω-periodic solution u for the difference system

$$D_2(x_t) = h(t)$$

is equivalent to the existence of continuous ω-periodic solution u for the difference system

$$[Ax](t) = h(t). \tag{4}$$

Therefore, for the sake of simplicity, we will study existence of continuous ω-periodic solution for the difference system (4). Beside this, we will also study some properties of the continuous ω-periodic solution for the difference system (4). Since B is a real matrix, from the theory of Jordan's form, there exists an invertible complex matrix U such that

$$UBU^{-1} = E_\lambda = diag(J_1, J_2, ..., J_l) \tag{5}$$

is Jordan's normal matrix, where for each $i = 1, 2, ..., l$ the matrix

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

Jordan's block matrices, with $\sum_{i=1}^{l} n_i = n$ and the set $\{\lambda_i : i = 1, 2, ..., l\}$ is the set of eigenvalues of matrix B. Now for each $i = 1, 2, ..., l$, let's define the operators

$$A_i : P_\omega \to P_\omega \ ; \quad [A_i y](t) = y(t) - \lambda_i \Sigma_{k=1}^{m} y(t - \tau_k), \ t \in [0, \omega].$$

Lemma 2.1. *For each $i = 1, 2, ..., l$ the operators T_i defined by $T_i : P_\omega \to P_\omega$; $[T_i x](t) = \lambda_i \sum_{k=1}^{m} x(t - \tau_k)$ have the following properties:*

(1) $\|T_i\| \le m|\lambda_i|$,

(2) $\int_0^\omega |[T_i x](t)|^p dt \le (m|\lambda_i|)^p \int_0^\omega |x(t)|^p dt, \ p \ge 1.$

Proof. **(1)** The proof is similar.

(2) Since x is ω-periodic, we have

$$\left(\int_0^\omega |[T_i x](t)|^p dt\right)^{\frac{1}{p}} \quad \le \quad |\lambda_i| \sum_{k=1}^{m} \left(\int_{-\tau_k}^{\omega - \tau_k} |x(s)|^p ds\right)^{\frac{1}{p}}$$

$$= \quad m|\lambda_i| \; (\int_0^\omega |x(s)|^p ds)^{\frac{1}{p}}.$$

Lemma 2.2. *If $m|\lambda_i| < 1$ then the inverse of A_i exists and its inverse $A_i^{-1} : P_\omega \to P_\omega$ satisfies the following properties:*

(1)$\|A_i^{-1}\| \leq \frac{1}{1-m|\lambda_i|}$,

(2)$\int_0^\omega |[A_i^{-1}f](t)|^p dt \leq (\frac{1}{1-m|\lambda_i|})^p \int_0^\omega |f(t)|^p dt$.

Proof. **(1)** Since $m|\lambda_i| < 1$, then $\|T_i\| < 1$. Therefore the operator $A_i = I - T_i$ has an inverse A_i^{-1} and the equality $\|A_i^{-1}\| \leq \frac{1}{1-m|\lambda_i|}$ is satisfied.

(2) Since $\|T_i\| < 1$ we can write

$$(\int_0^\omega |[A_i^{-1}f](t)|^p dt)^{\frac{1}{p}} = (\int_0^\omega |\sum_{j=0}^{\infty}[T_i^j f](t)|^p dt)^{\frac{1}{p}}.$$

Using the fact that

$$(\int_0^\omega |\sum_{j=0}^{\infty}[T_i^j f](t)|^p dt)^{\frac{1}{p}} \leq \sum_{j=0}^{\infty}(\int_0^\omega |[T_i^j f](t)|^p dt)^{\frac{1}{p}},$$

we obtain the inequality

$$(\int_0^\omega |[A_i^{-1}f(t)]|^p dt)^{\frac{1}{p}} \leq \sum_{j=0}^{\infty}(\int_0^\omega |[T_i^j f](t)|^p dt)^{\frac{1}{p}}.$$

Beside these, if we use the inequality

$$\begin{aligned} (\int_0^\omega |[T_i^j f](t)|^p dt)^{\frac{1}{p}} &\leq m|\lambda_i|(\int_0^\omega |[T_i^{j-1} f](t)|^p dt)^{\frac{1}{p}} \\ &\leq \cdots \leq (m|\lambda_i|)^j (\int_0^\omega |f(t)|^p dt)^{\frac{1}{p}} \end{aligned}$$

then we get

$$\begin{aligned} \sum_{j=0}^{\infty}(\int_0^\omega |[T_i^j f](t)|^p dt)^{\frac{1}{p}} &\leq \sum_{j=0}^{\infty}(m|\lambda_i|)^j (\int_0^\omega |f(t)|^p dt)^{\frac{1}{p}} \\ &= \frac{1}{1-m|\lambda_i|}(\int_0^\omega |f(t)|^p dt)^{\frac{1}{p}}. \end{aligned}$$

Therefore, the integral inequality

$$\int_0^\omega |[A_i^{-1}f](t)|^p dt \le (\frac{1}{1-m|\lambda_i|})^p \int_0^\omega |f(t)|^p dt$$

is obtained. □

Lemma 2.3. *If $\alpha \ge 1$, $\alpha \in \mathbb{R}$ then for $x, y \ge 0$, $x^\alpha + y^\alpha \le (x+y)^\alpha$.*

Theorem 2.4. *Let the matrix U and the operator A be as in defined in (5) and (3) respectively. If for any $i = 1, 2, ..., l$, $m|\lambda_i| < 1$, then the operator A has the inverse $A^{-1} : C_\omega \to C_\omega$ with the following properties:*

(1) $\|A^{-1}\| \le |U||U^{-1}|\sigma_0$, *where*

$$\sigma_0 = \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} (\frac{1}{1-m|\lambda_i|})^k m^{k-1}),$$

(2) *For any $f \in C_\omega$, $p \ge 1$ the inequality*

$$\int_0^\omega |[A^{-1}f](s)|^p ds \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |f(s)|^p ds$$

is satisfied. Here the constant σ_1 is given by

$$\sigma_1 = \begin{cases} [2^{n-1} \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} (\frac{1}{1-m|\lambda_i|})^k m^{k-1})^2]^{\frac{1}{2}}, & p = 1, \\ n^{\frac{2-p}{2}} [\sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} (\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}}, & p \in (1,2), \\ \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} (\frac{1}{1-m|\lambda_i|})^k m^{k-1})^2, & p = 2, \\ [\sum_{i=1}^{l} \sum_{j=1}^{n_i} (\sum_{k=1}^{j} (\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}}, & p \in (2,\infty), \end{cases}$$

and $q > 1$ is any constant satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Take $f \in C_\omega$ and consider the difference system

$$x(t) - B \sum_{k=1}^{m} x(t-\tau_k) = f(t). \tag{6}$$

In the Eq.(6), for $y \in T_\omega$ let's substitute $x(t) = U^{-1}y(t)$ then we obtain

$$y(t) - E_\lambda \sum_{k=1}^{m} y(t-\tau_k) = Uf(t) = g(t). \tag{7}$$

Therefore $g \in T_\omega$, (see Theorem 2.1 in [3] for form of functions f and g) Using Eq. (7), for $j = 1, 2, ..., n_i - 1$ and $i = 1, 2, ..., l$ we obtain the equations

$$y_{i,n_i}(t) - \lambda_i \sum_{k=1}^{m} y_{i,n_i}(t - \tau_k) = g_{i,n_i}(t), \tag{8}$$

$$y_{i,j}(t) - \lambda_i \sum_{k=1}^{m} y_{i,j}(t - \tau_k) - \sum_{k=1}^{m} y_{i,j+1}(t - \tau_k) = g_{i,j}(t). \tag{9}$$

If we apply Lemma 2.2 to Eq. (8), we obtain the inequations

$$|y_{i,n_i}|_\infty \leq \|A_i^{-1}\| |g_{i,n_i}|_\infty \leq \frac{1}{1 - m|\lambda_i|} |g_{i,n_i}|_\infty \tag{10}$$

and

$$\begin{aligned} (\int_0^\omega |y_{i,n_i}(t)|^p dt)^{\frac{1}{p}} &= (\int_0^\omega |[A_i^{-1} g_{i,n_i}](t)|^p dt)^{\frac{1}{p}} \\ &\leq \frac{1}{1 - m|\lambda_i|} (\int_0^\omega |g_{i,n_i}(t)|^p dt)^{\frac{1}{p}}. \end{aligned} \tag{11}$$

Moreover, as in Lemma 2.2 that the inverse of A_i exists and is bounded, then Eq.(8) has the unique solution $y_{i,n_i}(t)$. In the same way, by applying Lemma 2.2 to Eq.(9) for the case $j = n_i - 1$, we obtain that Eq.(9) also has a unique solution y_{n_i,n_i-1} and the following inequalities

$$|y_{i,n_i-1}|_\infty \leq \frac{1}{1 - m|\lambda_i|} |g_{i,n_i-1}|_\infty + (\frac{1}{1 - m|\lambda_i|})^2 m |g_{i,n_i}|_\infty \tag{12}$$

and

$$\begin{aligned} (\int_0^\omega |y_{i,n_i-1}(t)|^p dt)^{\frac{1}{p}}| \leq \frac{1}{1 - m|\lambda_i|} (\int_0^\omega |g_{i,n_i-1}(t)|^p dt)^{\frac{1}{p}} \\ + (\frac{1}{1 - m|\lambda_i|})^2 m (\int_0^\omega |g_{i,n_i}(t)|^p dt)^{\frac{1}{p}} \end{aligned} \tag{13}$$

are obtained. Later, for the other cases, for $y_{i,n_i-2}, y_{i,n_i-3}, ..., y_{i,2}, y_{i,1}$ if we use the same technique, the expressions

$$\begin{aligned} |y_{i,n_i-2}|_\infty \leq & \frac{1}{1 - m|\lambda_i|} |g_{i,n_i-2}|_\infty + (\frac{1}{1 - m|\lambda_i|})^2 m |g_{i,n_i-1}|_\infty \\ & + (\frac{1}{1 - m|\lambda_i|})^3 m^2 |g_{i,n_i}|_\infty, \end{aligned}$$

$$\vdots$$

$$\begin{aligned}|y_{i,2}|_\infty \leq & \frac{1}{1-m|\lambda_i|}|g_{i,2}|_\infty + (\frac{1}{1-m|\lambda_i|})^2 m|g_{i,3}|_\infty \\ & + \cdots + (\frac{1}{1-m|\lambda_i|})^{n_i-1} m^{n_i-2}|g_{i,n_i}|_\infty, \\ |y_{i,1}|_\infty \leq & \frac{1}{1-m|\lambda_i|}|g_{i,1}|_\infty + (\frac{1}{1-m|\lambda_i|})^2 m|g_{i,2}|_\infty \\ & + \cdots + (\frac{1}{1-m|\lambda_i|})^{n_i} m^{n_i-1}|g_{i,n_i}|_\infty,\end{aligned}$$

will be obtained. With together inequalities (10) and (12) if we sum up above inequalities, inequality

$$\sum_{j=1}^{n_i}|y_{i,j}|_\infty \leq \sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|g_{i,j}|_\infty \tag{14}$$

can be achieved. From the other side, if we sum up below inequalities

$$\begin{aligned}(\int_0^\omega |y_{i,n_i-2}(t)|^p dt)^{\frac{1}{p}} \leq & \frac{1}{1-m|\lambda_i|}(\int_0^\omega |g_{i,n_i-2}(t)|^p dt)^{\frac{1}{p}} \\ & + (\frac{1}{1-m|\lambda_i|})^2 m(\int_0^\omega |g_{i,n_i-1}(t)|^p dt)^{\frac{1}{p}} \\ & + (\frac{1}{1-m|\lambda_i|})^3 m^2(\int_0^\omega |g_{i,n_i}(t)|^p dt)^{\frac{1}{p}}, \\ & \vdots \\ (\int_0^\omega |y_{i,2}(t)|^p dt)^{\frac{1}{p}} \leq & \frac{1}{1-m|\lambda_i|}(\int_0^\omega |g_{i,2}(t)|^p dt)^{\frac{1}{p}} \\ & + (\frac{1}{1-m|\lambda_i|})^2 m(\int_0^\omega |g_{i,3}(t)|^p dt)^{\frac{1}{p}} + \cdots \\ & + (\frac{1}{1-m|\lambda_i|})^{n_i-1} m^{n_i-2}(\int_0^\omega |g_{i,n_i}(t)|^p dt)^{\frac{1}{p}}, \\ (\int_0^\omega |y_{i,1}(t)|^p dt)^{\frac{1}{p}} \leq & \frac{1}{1-m|\lambda_i|}(\int_0^\omega |g_{i,1}(t)|^p dt)^{\frac{1}{p}} \\ & + (\frac{1}{1-m|\lambda_i|})^2 m(\int_0^\omega |g_{i,2}(t)|^p dt)^{\frac{1}{p}} + \cdots \\ & + (\frac{1}{1-m|\lambda_i|})^{n_i} m^{n_i-1}(\int_0^\omega |g_{i,n_i}(t)|^p dt)^{\frac{1}{p}},\end{aligned}$$

with the inequalities (11) and (13), the inequality

$$\sum_{j=1}^{n_i}(\int_0^{\omega}|y_{i,j}(t)|^p dt)^{\frac{1}{p}} \le \sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})(\int_0^{\omega}|g_{i,j}(t)|^p dt)^{\frac{1}{p}}$$

is obtained.

(1) Now let's prove the first part of the theorem. Since the equations (8) and (9) have unique solutions then also the equation $y(t)-E_\lambda\sum_{k=1}^{m}y(t-\tau_k)=g(t)$ has the unique solution $y\in T_\omega$ and the norm of y satisfies

$$|y|_{T_\omega} \le (\sum_{i=1}^{l}\sum_{j=1}^{n_i}|y_{i,j}|_\infty^2)^{\frac{1}{2}} \le \sum_{i=1}^{l}\sum_{j=1}^{n_i}|y_{i,j}|_\infty.$$

Using (14), we have

$$\begin{aligned}|y|_{T_\omega} &\le \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|g_{i,j}|_\infty\\ &\le \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|g|_{T_\omega}\\ &= \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|Uf|_{T_\omega}\\ &\le \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|U||f|_{C_\omega}.\end{aligned}$$

Therefore, the difference system $x(t)-B\sum_{k=1}^{m}x(t-\tau_k)=f(t)$ has a unique periodic continuous solution $x(t)=U^{-1}y(t)$ which satisfies the inequality

$$\max_{t\in[0,\omega]}|x(t)| \le |U||U^{-1}|\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|f|_{C_\omega}.$$

As a result, we obtain that

$$|A^{-1}f|_{C_\omega} \le |U||U^{-1}|\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})|f|_{C_\omega}$$

that is

$$\|A^{-1}\| \le |U||U^{-1}|\sigma_0.$$

(2) For the second part of the theorem, we will give the proof for the different values of p.
If $\mathbf{p = 1}$; the proof of $p = 1$ is easily verified.
If $\mathbf{p \in (1, 2)}$; If we apply Lemma 2.3 and Minkowski inequality, we get

$$\begin{aligned}(\int_0^\omega |y(t)|^p dt)^{\frac{1}{p}} &\leq (\int_0^\omega (\sum_{i=1}^{l}\sum_{j=1}^{n_i} |y_{i,j}(t)|)^p dt)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\int_0^\omega |y_{i,j}(t)|^p dt)^{\frac{1}{p}},\end{aligned}$$

then using Hölder inequality,

$$\begin{aligned}(\int_0^\omega |y(t)|^p dt)^{\frac{1}{p}} &\leq \sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})(\int_0^\omega |g_{i,j}(t)|^p dt)^{\frac{1}{p}} \\ &\leq [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{1}{q}}[\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\int_0^\omega |g_{i,j}(t)|^p dt)]^{\frac{1}{p}} \\ &\leq [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{1}{q}}[\int_0^\omega \sum_{i=1}^{l}\sum_{j=1}^{n_i} |g_{i,j}(t)|^p dt]^{\frac{1}{p}}\end{aligned}$$

that is the inequality

$$\int_0^\omega |y(t)|^p dt \leq [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega \sum_{i=1}^{l}\sum_{j=1}^{n_i} |g_{i,j}(t)|^p dt \quad (15)$$

is obtained. If the inequality

$$\begin{aligned}\int_0^\omega \sum_{i=1}^{l}\sum_{j=1}^{n_i} |g_{i,j}(t)|^p dt &\leq n^{\frac{2-p}{2}} \int_0^\omega (\sum_{i=1}^{l}\sum_{j=1}^{n_i} |g_{i,j}(t)|^2 dt)^{\frac{p}{2}} \\ &= n^{\frac{2-p}{2}} \int_0^\omega |g(t)|^p dt\end{aligned}$$

is substituted into (15) then, we found

$$\int_0^\omega |y(t)|^p dt \leq n^{\frac{2-p}{2}} [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega |g(t)|^p dt.$$

Therefore for $p \in (1,2)$ the proof is completed.

If $\mathbf{p=2}$, the proof of the $p=2$ is similar to that of $p=1$. Here If we use Hölder inequality, then the inequality can be obtained. The proof is left to the researcher.

If $\mathbf{p} \in (\mathbf{2},\infty)$, if we use inequality (15) and Lemma 2.3 then we can write

$$\int_0^\omega |y(t)|^p dt$$
$$\le [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega \sum_{i=1}^{l}\sum_{j=1}^{n_i}[|g_{i,j}(t)|^2]^{\frac{p}{2}} dt$$
$$\le [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega [\sum_{i=1}^{l}\sum_{j=1}^{n_i}|g_{i,j}(t)|^2]^{\frac{p}{2}} dt$$
$$\le [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega |g(t)|^p dt.$$

As a result, inequality

$$\int_0^\omega |[A^{-1}f](t)|^p dt$$
$$\le |U^{-1}|^p [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega |g(t)|^p dt$$
$$= |U^{-1}|^p |U|^p [\sum_{i=1}^{l}\sum_{j=1}^{n_i}(\sum_{k=1}^{j}(\frac{1}{1-m|\lambda_i|})^k m^{k-1})^q]^{\frac{p}{q}} \int_0^\omega |f(t)|^p dt$$

can be obtained. □

Theorem 2.5. *Assume that $f \in C_\omega^1$ and for any $i = 1,2,\dots,l$ and $m|\lambda_i| < 1$. Then the inverse of A, $A^{-1} : C_\omega \to C_\omega$ satisfies the following properties:*

(1) $A^{-1}f \in C_\omega^1$;

(2) $[A^{-1}f]'(t) = [A^{-1}f'](t), \quad \forall t \in \mathbb{R}$;

(3) $\int_0^\omega |[A^{-1}f]'(t)|^p dt \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |f'(t)|^p dt, \ \ p \in [1,\infty)$.

Proof. **(1)** Since $f \in C_\omega^1$ and $g(t) = Uf(t)$ then $g \in T_\omega^1$. For an arbitrary positive integer n_i, from (8) and Lemma 2.2 since we have

$$[A_i y_{i,n_i}](t) = y_{i,n_i}(t) - \lambda_i \sum_{k=1}^{m} y_{i,n_i}(t-\tau_k) = g_{i,n_i}(t)$$

then we obtain

$$y_{i,n_i}(t) = [A_i^{-1} g_{i,n_i}](t) = [(I - T_i)^{-1} g_{i,n_i}](t) = \sum_{j=0}^{\infty} [T_i^j g_{i,n_i}](t),$$

that is we obtain $y_{i,n_i}(t) = \sum_{j=0}^{\infty} [T_i^j g_{i,n_i}](t)$. Since $g_{i,n_i} \in P_\omega^1$ and the infinite series $\sum_{j=0}^{\infty} [T_i^j g_{i,n_i}]'(t)$ is uniformly convergent on $\mathbb{R}$, then we have

$$y'_{i,n_i}(t) = \Big[\sum_{j=0}^{\infty} [T_i^j g_{i,n_i}](t) \Big]' = \sum_{j=0}^{\infty} [T_i^j g_{i,n_i}]'(t).$$

Therefore $y_{i,n_i} \in P_\omega^1$. In a similar way, using the Eq.(9). It can be shown that $y_{i,n_i-1}, y_{i,n_i-2}, ..., y_{i,1} \in P_\omega^1$. Thus $y \in T_\omega^1$. Beside, since $x(t) = U^{-1} y(t)$ then we have $A^{-1} \in C_\omega^1$.

(2) Now let's consider the system $x(t) - B \sum_{k=1}^{m} x(t - \tau_k) = f(t)$. This difference system has the unique solution $x = A^{-1} f \in C_\omega^1$. Therefore, in the difference system if we substitute $x = A^{-1} f$, then we obtain

$$[A^{-1} f](t) - B \sum_{k=1}^{m} [A^{-1} f](t - \tau_k) = f(t).$$

Beside, since f and $A^{-1} \in C_\omega^1$ then

$$[A^{-1} f]'(t) - B \sum_{k=1}^{m} [A^{-1} f]'(t - \tau_k) = f'(t).$$

Therefore, for any $t \in \mathbb{R}$, we obtain the result $[A^{-1} f]'](t) = [A^{-1} f'](t)$.

(3) In the 2nd condition of Theorem 2.4, instead of A^{-1} if we use $A^{-1} f'$ then in a similar manner, inequality

$$\int_0^\omega |[A^{-1} f]'(t)|^p dt \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |f'(t)|^p dt, \ p \in [1, \infty)$$

can be obtained. □

Remark 2.6. *Under the assumption that $m|\lambda_i| < 1$, $i = 1, 2, ..., l$ for any $k > 1$ positive integer if $f \in C^k(\mathbb{R}, \mathbb{R}^n) \cap C_\omega$ then like in Theorem 2.5 the following can be proved.*

(1) $\frac{d^k}{dt^k} [A^{-1} f](t) = [A^{-1} \frac{d^k}{dt^k} f](t)$*;*

(2) $\int_0^\omega |\frac{d^k}{dt^k}[A^{-1}f](t)|^p dt \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |\frac{d^k}{dt^k} f(t)|^p dt,\ p \in [1, \infty).$

Corollary 2.7. *If for $i = 1, 2, ..., l$ and $m|\lambda_i| < 1$ then the following holds:*

(1) *For any $f \in C_\omega$, the difference system*

$$D_2(x_t) = f(t) \tag{16}$$

has a unique ω-periodic solution and the inequality,

$$\int_0^\omega |x(t)|^p dt \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |f(t)|^p dt,\ p \in [1, \infty)$$

is satisfied.

(2) *Let $k > 1$ be a positive integer. Then for any $f \in C_\omega^k := C_\omega \cap C^k(\mathbb{R}, \mathbb{R}^n)$ the difference system (16) has a unique ω-periodic solution $x \in C_\omega^k$ such that it satisfies the inequality*

$$\int_0^\omega |x^{(k)}(t)|^p dt \le |U|^p |U^{-1}|^p \sigma_1 \int_0^\omega |f^{(k)}(t)|^p dt, p \in [1, \infty).$$

Theorem 2.8. *Assume that the operator D_2 is given by $D_2\varphi = \varphi(0) - B\sum_{k=1}^m \varphi(-\tau_k)$ and for any $i = 1, 2, ..., l$ and $m|\lambda_i| < 1$ the any ω-periodic solution x of Eq.(2) is continuously differentiable on $\mathbb{R}$.*

Proof. Let u be any ω-periodic solution of Eq.(2). Then the equality

$$\frac{d}{dt} D_2(u_t) = f(t, u_t)$$

is satisfied. If we take the integral over the interval $[0, \omega]$ then we have

$$\int_0^\omega f(s, u_s) ds = 0.$$

Beside, if we take the integral for $t \in [0, \omega]$, then $D_2(u_t) - D_2(u_0) = \int_0^t f(s, u_s) ds$ that is $D_2(u_t) = D_2(u_0) + \int_0^t f(s, u_s) ds$. So that the equation

$$D_2(u_t) = u(0) - B\sum_{k=1}^m u(-\tau_k) + \int_0^t f(s, u_s) ds$$

is obtained. Meanwhile, let

$$h(t) := u(0) - B\sum_{k=1}^{m} u(-\tau_k) + \int_0^t f(s, u_s)ds,$$

that is $D_2(u_t) = h(t)$. Since $f \in C((\mathbb{R} \times C[-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$ then $h \in C^1_\omega$. From the Corollary 2.7 we get the result $u \in C^1_\omega$. □

Corollary 2.9. *Since $m|\lambda_i| < 1$ the D_2-operator defined is stable. So that we show that when D_2-operator is stable, any ω-periodic solution of neutral functional differential equation (2) with multiple deviating arguments is continuously differentiable on $\mathbb{R}$. So that we generalized the study of (Lu et al.[3]) to neutral functional differential equation with multiple deviating arguments while D_2 is stable. But in this technique, we couldn't generalize the study of (Lu et al.[3]) to neutral functional differential equation with multiple deviating arguments while $m|\lambda_i| > 1$ that is D_2 is not stable. To overcome this difficulty, in the next section we will approach the problem with different way.*

3. A Different Approach to Linear Difference Operators with Multiple Deviating Arguments

In this part, we investigated the problem of existence of periodic solution for Eq.(2). In [4], the authors studied neutral functional differential system while D_1 is nonlinear operator. They obtained that even if $h \equiv 0$, the D_1-operator is not required to be stable. Note that operator D_2 defined in (2) is a linear operator. Now, we will approach the problem of existence of periodic solution of neutral functional differential equations with multiple deviating arguments while $m|\lambda_i| \neq 1$.

Let's write D_2-operator given in the previous section as

$$D_2(\varphi) = \varphi(0) - B\varphi(-\tau_1) - B\sum_{k=2}^{m} \varphi(-\tau_k).$$

Then from the definition of D_2 operator, we can write

$$D_2(x_t) = x(t) - Bx(t - \tau_1) - B\sum_{k=2}^{m} x(t - \tau_k).$$

Then, let's define the following operators:

$$T : C_\omega \to C_\omega \quad [Tx](t) := x(t) - Bx(t - \tau_1)$$

$$L : C_\omega \to C_\omega \quad [Lx](t) := B \sum_{k=2}^{m} x(t - \tau_k)$$

and

$$\Gamma : C_\omega \to C_\omega \quad [\Gamma x](t) := [Tx](t) - [Lx](t).$$

Then, form the definition of D_2-operator, for any $e \in C_\omega$, the existence of ω-periodic u solution of difference system

$$D_2(x_t) = e(t) \tag{17}$$

is equivalent to existence of ω-periodic u solution of difference system

$$[\Gamma x](t) = e(t). \tag{18}$$

As we discussed before, to indicate that Eq. (17) has a ω-periodic continuous solution. It is enough to show that Eq. (18) has a ω-periodic continuous solution.

Theorem 3.1. *Assume that the following hold:*

(1) *The functional L satisfied $L(0) = 0$ and for all $\varphi_1, \varphi_2 \in C_\omega$ there exists $\rho \in (0, \infty)$ such that*

$$|L(\varphi_1) - L(\varphi_2)| \leq \rho\, max_{t\in[0,\omega]} |\varphi_1(t) - \varphi_2(t)|;$$

(2) *For any $i = 1, 2, ..., l$ and $|\lambda_i| \neq 1$;*

(3) *$\rho |U^{-1}||U| \sum_{i=1}^{l} \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{1}{|1-|\lambda_i||^k} < 1$, then for any $e \in C_\omega$, the equation $D_2(x_t) = e(t)$ has a unique solution u^* in C_ω such that the inequalities*

$$\|u^*\| \leq \frac{|U^{-1}||U|\sigma}{1 - \rho |U^{-1}||U|\sigma} \|e\|$$

and

$$\|\Gamma^{-1}\| \leq \frac{|U^{-1}||U|\sigma}{1 - \rho |U^{-1}||U|\sigma}$$

are satisfied. Here, the constant $\sigma = \sum_{i=1}^{l} \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{1}{|1-|\lambda_i||^k}$.

Proof. In the proof of this theorem, we will use the Banach fixed point theorem. C_ω is a Banach space and

$$F : C_\omega \to C_\omega, \quad [Fx](t) = T^{-1}[Lx + e](t), \forall t \in [0, \omega]$$

is given. Note that there exists inverse of Toperator, (See Theorem 2.1 in [3]). From the definition of the operators T^{-1} and L, it is clear that $F(C_\omega) \subseteq C_\omega$. Now, let's show that F is a contraction operator. Take an arbitrary $x, y \in C_\omega$, then we have

$$\begin{aligned} \|Fx - Fy\| &= max_{t\in[0,\omega]}\big|[T^{-1}(Lx+e)](t) - [T^{-1}(Ly+e)](t)\big| \\ &\leq \|T^{-1}\|\rho\, max_{t\in[0,\omega]}|x(t)-y(t)| \\ &\leq \|T^{-1}\|\rho\,\|x-y\|. \end{aligned}$$

But, by the 3rd condition we have

$$\rho|U^{-1}||U|\sum_{i=1}^{l}\sum_{j=1}^{n_i}\sum_{k=1}^{j}\frac{1}{|1-|\lambda_i||^k} < 1$$

then by the Banach fixed point theorem F has a unique fixed point $u^* \in C_\omega$. Therefore, $[Tu^*](t) = [Lu^*](t) + e(t)$, $\forall t \in \mathbb{R}$ and thus u^* is a unique ω-periodic solution of $[\Gamma u^*](t) = e(t)$, $\forall t \in \mathbb{R}$. But, this is equivalent that the difference system $D_2(x_t) = e(t)$, $\forall t \in \mathbb{R}$ has a unique u^* solution in C_ω. From the other side, we have

$$\begin{aligned} \|u^*\| &= max_{t\in[0,\omega]}|u^*(t)| = max_{t\in[0,\omega]}|T^{-1}[Lu^*+e](t)| \\ &= max_{t\in[0,\omega]}|[T^{-1}Lu^*](t) - [T^{-1}e](t)| \\ &\leq \|T^{-1}\|\big(max_{t\in[0,\omega]}|Lu^*(t)| + max_{t\in[0,\omega]}|e(t)|\big) \end{aligned}$$

But using first condition of Theorem 3.1, we have $|L(u^*)| = |L(u^*) - L(0)| \leq \|u^* - 0\|$, then we obtain that $\|u^*\| \leq \|T^{-1}\|\rho\|u^*\| + \|T^{-1}\|\|e\|$. But using Theorem 2.1 in [3], we get

$$\|u^*\| \leq \frac{|U^{-1}||U|\sigma}{1-\rho|U^{-1}||U|\sigma}\|e\|.$$

But since $u^*(t) = [\Gamma^{-1}e](t)$ then the result $\|\Gamma^{-1}\| \leq \frac{|U^{-1}||U|\sigma}{1-\rho|U^{-1}||U|\sigma}\|e\|$ is obtained. □

Remark 3.2 ([5]). *As $f \in P_\omega$, $f(t) = \sum_{n\in\mathbb{Z}} f_n e^{i\mu_n t}$, where the coefficients $f_n = \frac{1}{\omega}\int_0^\omega f(s)e^{-i\mu_n s}ds$ are fourier coefficients, $\mu_n = \frac{2n\pi}{\omega}$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R}$, $\mathbb{Z}$ is the set of integers. Then if the operator E is defined by $E : P_\omega \to P_\omega$; $[Ex](t) = x(t) - \lambda\sum_{k=1}^{m} x(t-\tau_k)$ then for any $f \in P_\omega$, if $[Ex](t) = f(t)$ thus we have*

$$[E^{-1}f](t) = \sum_{n\in\mathbb{Z}} \frac{f_n}{1-\lambda\sum_{k=1}^{m} e^{-i\mu_n\tau_k}} e^{i\mu_n t}.$$

Theorem 3.3 ([6]). *Suppose that $\phi \in C^k(\mathbb{R},\mathbb{R})$ and for $n \in \mathbb{Z}$, $\mu_n = \frac{2n\pi}{\omega}$, $c_n = \frac{1}{\omega}\int_0^\omega \phi(s)e^{-\mu_n s}ds$ are the Fourier coefficients of ϕ then for $k \geq 2$ the Fourier series $\sum_{n\in\mathbb{Z}} c_n e^{i\mu_n t}$ is uniformly convergent.*

Theorem 3.4. *Suppose that the conditions (1)-(3) of Theorem 3.1 hold, and for $k \geq 2$, $f \in C_\omega^k$. Therefore, the inverse of Γ, $\Gamma^{-1} : C_\omega \to C_\omega$ exists and satisfies the following properties:*

(1) $\Gamma^{-1} f \in C_\omega^k$,

(2) $[\Gamma^{-1} f]^{(k)}(t) = [\Gamma^{-1} f^{(k)}](t), \forall t \in \mathbb{R}$.

Proof. **(1)** Existence of Γ^{-1} follows from Theorem 3.1. Since $g(t) = Uf(t)$ then $g \in T_\omega^k$ and from Eq.(8) we can write

$$[A_i^{-1} g_{i,n_i}](t) = y_{i,n_i}(t) = \sum_{n\in\mathbb{Z}} \frac{g_n}{1-\lambda_i \sum_{k=1}^m e^{-i\mu_n \tau_k}} e^{i\mu_n t}.$$

Here $\frac{g_n}{1-\lambda_i \sum_{k=1}^m e^{-i\mu_n \tau_k}} = \frac{1}{\omega}\int_0^\omega [A_i^{-1} g_{i,n_i}](s)e^{-i\mu_n s}ds$, $\mu_n = \frac{2n\pi}{\omega}$, $n \in \mathbb{Z}$. Meanwhile, Since the series

$$y_{i,n_i}(t) = \sum_{n\in\mathbb{Z}} \frac{g_n}{1-\lambda_i \sum_{k=1}^m e^{-i\mu_n \tau_k}} e^{i\mu_n t},$$

$$y_{i,n_i}^{(k)}(t) = \sum_{n\in\mathbb{Z}} \frac{g_n}{1-\lambda_i \sum_{k=1}^m e^{-i\mu_n \tau_k}} (i\mu_n)^k e^{i\mu_n t}$$

are uniformly convergent, then $y_{i,n_i} \in P_\omega^k$. Similarly, using Eq.(9), it can be shown that $y_{i,n_i-1}, y_{i,n_i-2}, ..., y_{i,1} \in P_\omega^k$. As a result $y \in T_\omega^k$, since $x = U^{-1}y$ and so $\Gamma^{-1} f = x \in C_\omega^k$.

(2) Into the equation $[\Gamma x](t) = x(t) - B\sum_{k=1}^m x(t-\tau_k) = f(t)$ if we substitute $[\Gamma^{-1} f](t) = x(t)$, then we obtain

$$[\Gamma^{-1} f](t) - B\sum_{k=1}^m [\Gamma^{-1} f](t-\tau_k) = f(t)$$

but from the 1st property $\Gamma^{-1} f \in C_\omega^k$ therefore, we get

$$[\Gamma^{-1} f]^{(k)}(t) - B\sum_{k=1}^m [\Gamma^{-1} f]^{(k)}(t-\tau_k) = f^{(k)}(t),$$

that is, $[\Gamma^{-1} f^{(k)}](t) = [\Gamma^{-1} f]^{(k)}(t)$.

Remark 3.5. *In this last part of third section, using series of Fourier we showed that any ω-periodic solution x for neutral functional difference equation $\frac{d}{dt}D_2(x_t) = f(t, x_t)$ is continuously differentiable on $\mathbb{R}$, while $m|\lambda_i| > 1$ that is in the event of operator D_2 is unstable.*

4. Applications

In this section, we will apply the properties of D_2-operator obtained in Section 2 to study the existence of periodic solution for Eq.(2). In order to do this, we recall Mawhin's continuation theorem in the first.

Theorem 4.1 ([7]). *Let X and Y be two Banach spaces, $L : Dom(L) \subset X \to Y$ be an Fredholm operator of index zero, $\Omega \subset X$ be an open, bounded subset of X with $Dom(L) \cap \Omega \neq \emptyset$ and $N : \overline{\Omega} \to Y$ be an L-compact operator on $\overline{\Omega}$. Suppose that the following conditions hold:*

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega$, $\forall \lambda \in (0, 1)$,

(2) $Nx \notin ImL$, $\forall x \in \partial\Omega \cap KerL$,

(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$

Where $J : ImQ \to KerL$ is an isomorphism. Then operator equation $Lx \neq Nx$ has at least one solution on $\overline{\Omega} \cap Dom(L)$.

Now, using Theorem 4.1 and properties of D_2-operator we will investigate periodic solution of the equation $\frac{d}{dt}D_2(x_t) = f(t, x_t)$. Take $Dom(L) = \{y \in C_\omega : D_2 y_t \text{ is continuously differentiable over} \, \mathbb{R}\}$ and $X = Y = C_\omega$. Then define

$$L : Dom(L) \cap C_\omega \to C_\omega,$$
$$[Lx](t) = \frac{d}{dt}D_2(x_t) = \frac{d}{dt}[x(t) - B(x(t - \tau_1) + x(t - \tau_2))]$$

and

$$N : C_\omega \to C_\omega; \; [Ly](t) = f(t, y_t).$$

Since $KerL \cong \mathbb{R}^n$ we have $dim(kerL) = n$. $CoKerL \cong ImQ$ and from the definition of Q, $ImQ \cong \mathbb{R}^n$. Therefore $CodimImL = dimCokerL = n$. Hence, using also ImL is a closed set in C_ω and $ImL = \left\{y \in C_\omega : \int_0^\omega y(s)ds = 0\right\}$ we see that $L : Dom(L) \subset X \to Y$ is a Fredholm operator of index zero. Now define

$$K_p : ImL \subset C_\omega \to Dom(L) \cap KerP; \; [K_p z](t) = [A^{-1}Fz](t)$$

with $[Fz](t) = \int_0^t z(s)ds$. From Theorem 2.5 we have $A^{-1}Fz \in C_\omega^1$, so that K_p is completely continuous, therefore N is L-compact operator on $\overline{\Omega}$.

Theorem 4.6. *Suppose $m|\lambda_i| < 1$ for all $i = 1, 2, ..., l$ and $\Omega \subset C_\omega$ is an open bounded set. If all of the following conditions hold:*

(1) For each $\lambda \in (0,1)$, the equation $\frac{d}{dt}D_2(x_t) = \lambda f(t, x_t)$ has no solution on $\partial\Omega$.

(2) The equation $QNx = \frac{1}{\omega}\int_0^\omega f(s, x_s)ds = 0$ has no solution on $\partial\Omega \cap \mathbb{R}^n$.

(3) The Brouwer degree $d_B\{QN, \partial\Omega \cap \mathbb{R}^n, 0\} \neq 0$.

Then Eq. (2) has at least one ω-periodic solution in $\overline{\Omega}$.

Example 4.7. Consider the equation

$$\frac{d}{dt}\big(x(t) - B[x(t-\tau_1) + x(t-\tau_2)]\big) = \sigma g(x(t-\mu)) + e(t) \tag{19}$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, σ is a positive constant, $g(x) = (\frac{x_1^3}{1+x_1^2} + \frac{x_2^5}{2+x_2^4}, \frac{3x_1^2}{1+x_1^2} + \frac{x_2^5}{2+x_2^4})^T$, $B = \begin{pmatrix} \frac{1}{3} & -\frac{4}{9} \\ 0 & -\frac{1}{3} \end{pmatrix}$, and $e(t) = (sin(t), cos(t))$. Corresponding to Eq.(2), $f(t, x_t) = \sigma g(x_t(\mu)) + e(t) = \sigma g(x(t-\mu)) + e(t)$. Also, we can choose $U = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{pmatrix}$ and $U^{-1} = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$ such that $UBU^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$, so we have the eigenvalues $\lambda_1 = \frac{1}{3}$, $\lambda_2 = -\frac{1}{3}$ with $2|\lambda_i| < 1$, $i = 1, 2$.

Assume that for $\lambda \in (0,1)$, $u \in C_{2\pi}$ be an arbitrary solution of the equation

$$\frac{d}{dt}(x(t) - B[x(t-\tau_1) + x(t-\tau_2)]) = \lambda\sigma g(x(t-\mu)) + \lambda e(t). \tag{20}$$

Therefore $\int_0^{2\pi} g(u(t-\mu))dt = -\int_0^{2\pi} e(t)dt$. So that $\int_0^{2\pi} g(u(t))dt = 0$. Hence, we have

$$\int_0^{2\pi}[\frac{u_1^3(t)}{1+u_1^2(t)} + \frac{u_2^5(t)}{2+u_2^4(t)}]dt = \int_0^{2\pi}[\frac{3u_1^2(t)}{1+u_1^2(t)} + \frac{u_2^5(t)}{2+u_2^4(t)}]dt = 0.$$

Thus

$$\int_0^{2\pi}\frac{u_1^2(t)[u_1(t)-3]}{1+u_1^2(t)} = 0 \text{ and } \int_0^{2\pi}[\frac{3u_1^2(t)}{1+u_1^2(t)} + \frac{u_2^5(t)}{2+u_2^4(t)}]dt = 0.$$

Therefore there exists $t_0, t_1 \in [0, 2\pi]$ such that $u_1(t_0) = 3$ and $|u_2(t_1)| \leq 1$ and the inequality

$$|u_1|_\infty \leq 3 + \int_0^{2\pi} |u_1^{'}(s)|ds \text{ and } |u_2|_\infty \leq 3 + \int_0^{2\pi} |u_2^{'}(s)|ds$$

are satisfied and from Theorem 2.8 $u \in C^1_{2\pi}$. Now, we will try to investigate an upper bound for $\int_0^{2\pi} |u^{'}(t)|dt$. Using Theorem 2.5 and Eq.(20) we have

$$\begin{aligned}\int_0^{2\pi} |u^{'}(t)|dt &= \int_0^{2\pi} |A^{-1}Au^{'}(t)|dt = \int_0^{2\pi} |A^{-1}Au(t)|^{'}dt \\ &\leq |U^{-1}||U|\sqrt{2}[\sum_{i=1}^{2}\sum_{j=1}^{n_i}\Big(\sum_{k=1}^{j}(\frac{1}{1-2|\lambda_i|})^k 2^{k-1}\Big)^2]^{\frac{1}{2}} \int_0^{2\pi} |[Au]^{'}(t)|dt \\ &\leq |U^{-1}||U|\sqrt{2}\Big(\frac{1}{1-2|\lambda_1|} + \frac{1}{1-2|\lambda_2|}\Big)\Big(\sigma \int_0^{2\pi} |g(x(t-\mu))|dt + 2\pi\Big).\end{aligned}$$

Since $\int_0^{2\pi} |g(x(t-\mu))|dt \leq 6\pi(|u_1|_\infty + |u_2|_\infty + 1)$, using this inequality we obtain

$$\begin{aligned}&\int_0^{2\pi} |u^{'}(t)|dt \\ &\quad\leq |U^{-1}||U|\sqrt{2}\Big(\frac{1}{1-2|\lambda_1|} + \frac{1}{1-2|\lambda_2|}\Big)\{6\pi\sigma(|u_1|_\infty + |u_2|_\infty + 1) + 2\pi\} \\ &\qquad\leq \frac{88\sqrt{2}\pi}{3}\{3\sigma(6 + \sqrt{2}\int_0^{2\pi} (|u_1^{'}(s)|^2 + |u_2^{'}(s)|^2)^{\frac{1}{2}}ds + 1) + 1\},\end{aligned}$$

that is

$$\begin{aligned}\int_0^{2\pi} |u^{'}(t)|dt &\leq \frac{88\sqrt{2}\pi}{3}\{3\sigma(7 + \sqrt{2}\int_0^{2\pi} |u^{'}(s)ds) + 1\} \\ &\leq \frac{88\sqrt{2}\pi}{3}\{21\sigma + 3\sqrt{2}\sigma\int_0^{2\pi} |u^{'}(s)ds + 3\sigma\} \\ &= 616\sqrt{2}\pi\sigma + 176\pi\sigma\int_0^{2\pi} |u^{'}(s)|ds + 88\sqrt{2}\pi\sigma\end{aligned}$$

for $\sigma < \frac{1}{176\pi}$. We have $\int_0^{2\pi} |u^{'}(t)|dt \leq \frac{616\sqrt{2}\pi\sigma + 88\sqrt{2\pi}}{1-176\pi\sigma} : M$. Hence, we find that

$$\begin{aligned}|u|_{C_{2\pi}} &\leq |u_1|_\infty + |u_2|_\infty \\ &\leq 6 + \int_0^{2\pi} (|u_1^{'}(t)| + |u_2^{'}(t)|)dt\end{aligned}$$

$$\begin{aligned}
&\leq 6+\sqrt{2}\int_0^{2\pi}(|u_1^{'}(t)|^2+|u_2^{'}(t)|^2)^{\frac{1}{2}}dt \\
&\leq 6+\sqrt{2}\int_0^{2\pi}|u^{'}(t)|dt \\
&= 6+\sqrt{2}M := M_0
\end{aligned}$$

and then in this way the set Ω can be defined by $\Omega = \{x \in C_{2\pi} : |x|_{C_{2\pi}} < M_0 + 1\}$. So that the first condition of Theorem 4.6 is satisfied.

Now, assume that for $u \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}^2$ we have $Nu \in ImL = \{y \in C_\omega : \int_0^\omega y(s)ds = 0\}$. That is, we have

$$\int_0^\omega Nu(s)ds = \int_0^\omega f(s,x_s)ds = \int_0^\omega (\sigma g(x(s-\mu)) + e(s))ds = 0$$

Since $\int_0^\omega e(s)ds = 0$ then $\sigma\int_0^\omega g(x(s-\mu))ds = 0$ and therefore $\int_0^\omega g(x(s))ds = 0$. Beside, since $u = (x_1,x_2)^T \in \mathbb{R}^2$ then we obtain the equation

$$\frac{x_1^3}{1+x_1^2}+\frac{x_2^5}{2+x_2^4}=0 \text{ and } \frac{3x_1^2}{1+x_1^2}+\frac{x_2^5}{2+x_2^4}=0.$$

The solution set of this system is $(x_1,x_2)^T = \{(3,-2.78)^T,(0,0)^T\}$.

Thus $(x_1,x_2)^T \notin \partial\Omega \cap KerL$, that is $\forall u \in \partial\Omega \cap KerL$, $Nu \notin ImL$. So the second condition of continuation theorem is satisfied.

Lastly, since $(JQN)^{-1}[(0,0)^T] = \{(3,-2.78)^T,(0,0)^T\}$ then

$$\begin{aligned}
d_B\{JQN,\Omega\cup\mathbb{R}^2,0\} &= sgn\ det\frac{\partial g(x)}{\partial x}_{x_1=3,\,x_2=-2.78} \\
&\quad + sgn\ det\frac{\partial g(x)}{\partial x}_{x_1=0,\,x_2=0} = 1 \neq 0
\end{aligned}$$

and then by Theorem 4.6 for $\sigma < \frac{1}{176\pi}$ the equation (19) has at least one 2π-periodic solution in $\overline{\Omega}$.

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