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EXISTENCE OF POSITIVE SOLUTION FOR A NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract: The existence of positive solution is obtained for the following nonlinear three-point boundary value problem

$$\begin{cases} u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1) \\ \beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

where

$$\beta, \gamma \geq 0, 0 < \eta < 1, 0 < \alpha < \frac{1}{\eta} \text{ and } d = \beta(1-\alpha\eta) + \gamma(1-\alpha) > 0.$$

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1. Introduction

Recently, multi-point boundary value problems have received much attention from many authors [2], 2–5. In particular, Ma [5] obtained the existence of one positive solution for the boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1)\,, \tag{1}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \tag{2}$$

where $0 < \eta < 1$ and $0 < \alpha < 1/\eta$, and Liu [4] discussed the existence of positive solutions to the boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \tag{3}$$

$$u'(0) = 0, \quad u(1) = \alpha u(\eta), \tag{4}$$

where $0 < \eta, \alpha < 1$. In the above papers [5, 4], the authors require that f is either superlinear or sublinear, and their main results are based upon an application of the well-known Krasnoselskii's Fixed Point Theorem [1].

The purpose of this paper is to study the following more general nonlinear three-point boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1) \tag{5}$$

$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta), \tag{6}$$

where $\beta, \gamma \geq 0, 0 < \eta < 1, 0 < \alpha < \frac{1}{\eta}$ and $d = \beta(1-\alpha\eta)+\gamma(1-\alpha) > 0$. In Section 3, we establish some existence results of positive solution for the problem (5) and (6), and we do not require that f is either superlinear or sublinear.

Throughout, we assume that the following conditions are satisfied:

(A1) $f \in C([0,\infty),[0,\infty))$.

(A2) $a \in C([0,1],[0,\infty))$ and there exists $x_0 \in [\eta,1]$ such that $a(x_0) > 0$.

(A3) $d > 0$.

2. Preliminary Lemmas

Lemma 2.1. *Let $d \neq 0$. Then for $h \in C[0,1]$, the problem*

$$u''(t) + h(t) = 0, \quad t \in (0,1)\,, \tag{7}$$

$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta)\,, \tag{8}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)h(s)ds + \frac{\beta t + \gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds\right].$$

Proof. From (7) and (8), it is easy to see that the result holds. □

Lemma 2.2. *Let $d > 0$. If $h \in C[0,1]$ and $h \geq 0$, then the unique solution u of the problem (7) and (8) satisfies $u(t) \geq 0$ for $t \in [0,1]$.*

Proof. From the fact that $u''(t) = -h(t) \leq 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$. So it suffices to prove that $u(0) \geq 0$ and $u(1) \geq 0$.

To show that $u(0) \geq 0$, there are two cases to be considered. We first consider the case $0 < \alpha < 1$. In this case

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds\right] \\ &\geq \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^1 (1 - s)h(s)ds\right] \\ &= \frac{\gamma(1-\alpha)}{d}\int_0^1 (1-s)h(s)ds \\ &\geq 0. \end{aligned}$$

Next we consider the case $\alpha \geq 1$. In this case

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right] \\ &= \frac{\gamma}{d}\left[\int_0^\eta \left[(1-\alpha\eta) + (\alpha - 1)s\right] h(s)ds + \int_\eta^1 (1-s)h(s)ds\right] \\ &\geq 0. \end{aligned}$$

Furthermore, we know that

$$\begin{aligned} u(1) &= -\int_0^1 (1-s)h(s)ds \\ &\quad + \frac{\beta+\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right] \\ &= \frac{\alpha(\beta\eta+\gamma)}{d}\int_0^1 (1-s)h(s)ds - \frac{\alpha(\beta+\gamma)}{d}\int_0^\eta (\eta-s)h(s)ds \\ &\geq \frac{\alpha(\beta\eta+\gamma)}{d}\int_0^\eta (1-s)h(s)ds - \frac{\alpha(\beta+\gamma)}{d}\int_0^\eta (\eta-s)h(s)ds \\ &= \frac{\alpha(1-\eta)}{d}\int_0^\eta (\beta s+\gamma)h(s)ds \geq 0. \end{aligned}$$

The proof is complete. □

Similar to the proof of Lemma 4 of [5], we can obtain the following lemma.

Lemma 2.3. *Let $d > 0$. If $h \in C[0,1]$ and $h \geq 0$, then the unique solution u of the problem (7) and (8) satisfies*

$$\min_{t\in[\eta,1]} u(t) \geq \sigma \left\|u\right\|,$$

where

$$\sigma = \min\left\{\frac{\alpha(1-\eta)}{1-\alpha\eta}, \alpha\eta, \eta\right\} \text{ and } \left\|u\right\| = \max_{t\in[0,1]} \left|u(t)\right|.$$

The following fixed point theorem (see [1]) is very crucial in our arguments.

Theorem 2.4. (Krasnoselskii's Fixed Point Theorem) *Let E be a Banach space, and K be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$\Phi : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2,$

or

(ii) $\|\Phi u\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|\Phi u\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2.$

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main Results

For convenience, we let

$$D = \frac{d}{\beta + \gamma} \left[\int_0^1 (1-s)a(s)ds \right]^{-1}$$

and

$$C = \frac{d}{\beta\eta + \gamma} \left[\int_\eta^1 (1-s)a(s)ds \right]^{-1}.$$

Theorem 3.1. *Assume that there exist two different positive numbers ν and μ such that*

$$f(u) \leq \nu D \text{ for } u \in [0, \nu] \tag{9}$$

and

$$f(u) \geq \mu C \text{ for } u \in [\sigma\mu, \mu]. \tag{10}$$

Then, the boundary value problem (5) and (6) has at least one positive solution u such that

$$\min\{\nu,\mu\} \le \|u\| \le \max\{\nu,\mu\}.$$

Proof. Set

$$K = \left\{ u \,\middle|\, u \in C[0,1], u \ge 0, \min_{t\in[\eta,1]} u(t) \ge \sigma \|u\| \right\}$$

and

$$\Phi u(t) = -\int_0^t (t-s)a(s)f(u(s))ds$$

$$+\frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)a(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)a(s)f(u(s))ds\right].$$

Then, it is obvious that K is a cone in $C[0,1]$. Moreover, by Lemma 2.3, $\Phi(K) \subset K$. It is also easy to check that $\Phi : K \to K$ is completely continuous and that u is a solution of the problem (5) and (6) if and only if u is a fixed point of Φ.

Without loss of generality, we may assume $\nu < \mu$.

Set

$$\Omega_1 = \{u \in C[0,1] \,|\, \|u\| < \nu\} \text{ and } \Omega_2 = \{u \in C[0,1] \,|\, \|u\| < \mu\}.$$

For $u \in K \cap \partial\Omega_1$, we have from (9) that

$$\begin{aligned}
\Phi u(t) &= -\int_0^t (t-s)a(s)f(u(s))ds \\
&\quad +\frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)a(s)f(u(s))ds\right. \\
&\quad \left.-\alpha\int_0^\eta (\eta-s)a(s)f(u(s))ds\right] \\
&\le \frac{\beta t+\gamma}{d}\int_0^1 (1-s)a(s)f(u(s))ds
\end{aligned}$$

$$\begin{aligned} &\leq \nu D \frac{\beta+\gamma}{d} \int_0^1 (1-s)a(s)ds \\ &= \nu = \|u\|, \qquad t \in [0,1], \end{aligned}$$

and so,

$$\|\Phi u\| \leq \|u\| \text{ for } u \in K \cap \partial\Omega_1. \tag{11}$$

For $u \in K \cap \partial\Omega_2$, we get from (10) that

$$\begin{aligned} \Phi u(\eta) &= -\int_0^\eta (\eta - s)a(s)f(u(s))ds \\ &\quad + \frac{\beta\eta+\gamma}{d} \left[\int_0^1 (1-s)a(s)f(u(s))ds \right. \\ &\quad \left. -\alpha \int_0^\eta (\eta-s)a(s)f(u(s))ds \right] \\ &\geq \frac{\beta\eta+\gamma}{d} \int_\eta^1 (1-s)a(s)f(u(s))ds \\ &\geq \mu C \frac{\beta\eta+\gamma}{d} \int_\eta^1 (1-s)a(s)ds \\ &= \mu = \|u\|. \end{aligned}$$

So,

$$\|\Phi u\| \geq \|u\| \text{ for } u \in K \cap \partial\Omega_2. \tag{12}$$

Therefore, it follows from (11), (12) and Theorem 2.4 that the boundary value problem (5) and (6) has at least one positive solution u such that

$$\nu \leq \|u\| \leq \mu.$$

The proof is complete. □

Corollary 3.1. *The boundary value problem (5) and (6) has at least one positive solution if either*

$$\overline{\lim}_{u\to 0}\frac{f(u)}{u} < D \text{ and } \underline{\lim}_{u\to\infty}\frac{f(u)}{u} > \frac{C}{\sigma}, \tag{13}$$

or

$$\underline{\lim}_{u\to 0}\frac{f(u)}{u} > \frac{C}{\sigma} \text{ and } \overline{\lim}_{u\to\infty}\frac{f(u)}{u} < D. \tag{14}$$

Proof. First, we suppose (13) holds.

Since $\overline{\lim}_{u\to 0}\frac{f(u)}{u} < D$, we may choose $\nu > 0$ such that

$$f(u) \le Du \le D\nu \text{ for } u \in [0,\nu]. \tag{15}$$

Furthermore, since $\underline{\lim}_{u\to\infty}\frac{f(u)}{u} > \frac{C}{\sigma}$, there exists $d > 0$ such that

$$f(u) \ge \frac{C}{\sigma}u \text{ for } u \ge d.$$

Let $\mu > \max\left\{\nu, \frac{d}{\sigma}\right\}$, so

$$f(u) \ge \frac{C}{\sigma}u \ge C\mu \text{ for } u \in [\sigma\mu,\mu]. \tag{16}$$

Therefore, by (15), (16) and Theorem 3.1, it follows that the boundary value problem (5) and (6) has at least one positive solution.

Next, we suppose (14) holds.

In view of $\underline{\lim}_{u\to 0}\frac{f(u)}{u} > \frac{C}{\sigma}$, we may choose $\mu > 0$ such that

$$f(u) \ge \frac{C}{\sigma}u \text{ for } u \in (0,\mu].$$

Especially, for $u \in [\sigma\mu,\mu]$, we have

$$f(u) \ge \frac{C}{\sigma}u \ge C\mu. \tag{17}$$

On the other hand, since $\overline{\lim}_{u\to\infty}\frac{f(u)}{u} < D$, there exists $d > 0$ such that

$$f(u) \le Du \text{ for } u \ge d. \tag{18}$$

We consider two cases:

Case (i). $\overline{\lim}_{u\to\infty} f(u) = +\infty$. In this case, there exist $c_n(n = 1, 2, \cdots) \to +\infty$ so that

$$f(c_n) = \max_{0\le u\le c_n} f(u). \tag{19}$$

Choose n_0 such that $\nu = c_{n_0} > \max\{\mu, d\}$, then by (18) and (19), we have

$$f(u) \le f(\nu) \le D\nu \text{ for } u \in [0, \nu]. \tag{20}$$

Case (ii). If $\overline{\lim}_{u\to\infty} f(u) < +\infty$, then f is bounded on $[0, +\infty)$. Let

$$M = \sup\{f(u)\,|u \in [0, +\infty)\},$$

and choose $\nu > \max\left\{\mu, \frac{M}{D}\right\}$, then we have

$$f(u) \le M \le D\nu \text{ for } u \in [0, +\infty).$$

Especially, we have

$$f(u) \le D\nu \text{ for } u \in [0, \nu]. \tag{21}$$

Therefore, we may get $\nu > \mu > 0$ such that

$$f(u) \le D\nu \text{ for } u \in [0, \nu]. \tag{22}$$

Then, by (17), (22) and Theorem 3.1, it follows that the boundary value problem (5) and (6) has at least one positive solution. The proof is complete. □

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