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STEADY STATE WITH SMALL CHANGE OF REPRODUCTION AND SELF-LIMITATION

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Abstract: In [8], we established some sufficient condition for the uniqueness of the positive solution to the general elliptic system for two competing species of animals

$$\begin{cases} \Delta u + u(g_1(u) - g_2(v)) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(h_1(v) - h_2(u)) = 0 & \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. & \end{cases}$$

In this paper, we try to extend the uniqueness result by perturbing the reproduction and self-limitation functions g_1, h_1 of the above model. The techniques used in this paper are super-sub solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations.

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1. Introduction

A lot of attention has been given to the following Lotka-Volterra system that models the competitive interaction of two species of animal residing in the same environment:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) & \\ & \text{in } \Omega \times R^+, \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) - eu(x,t)) & \\ u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0, & \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n. The solutions of (1) represent population densities for the competing species. The positive constant coefficients represent growth rates (a and d), self-limitation rates (b and f) and competition rates (c and e). There are several results for existence, uniqueness and stability of positive steady state solution to (1)(see [1], [2], [3], [4], [6], [7]), i.e. positive solution to

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 & \\ & \text{in } \Omega, \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 & \\ u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0. & \end{cases} \tag{2}$$

Now, we pay attention to some existence and uniqueness results that has already proven.

In 1984, Cosner and Lazer (see [4]) established the following sufficient conditions for existence and uniqueness of positive steady state solution to (2).

Theorem 1.1. (see [4]) *Suppose:*

(A) $a > \lambda_1 + \frac{cd}{f}, d > \lambda_1 + \frac{ae}{b}$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta$ *with homogeneous boundary condition,*

(B) $4bf > \frac{fc^2}{b} \sup_{x\in\Omega}[\frac{\omega_a(x)}{\omega_{d-\frac{ae}{b}}(x)}] + 2ce + \frac{be^2}{f} \sup_{x\in\Omega}[\frac{\omega_d(x)}{\omega_{a-\frac{cd}{f}}(x)}]$, *where* $\omega_M(x)$ *for* $M > 0$ *is the unique positive solution to the logistic equation as mentioned in the next section.*

Then (2) has a unique positive solution.

Theorem 1.1 implies that if the self-reproduction and self-limitation rates are relatively large, in other words, the competition rates are relatively small, then there is a unique positive steady state solution of (2).

Furthermore, in 1989, Cantrell and Cosner (see [3]) also proved that we can extend the region of reproduction and self-limitation rates without losing the uniqueness under certain conditions.

Theorem 1.2. (see [3]) *If $a = d > \lambda_1, b = f = 1$ and $0 < c, e < 1$, then there is a neighborhood V of (a, a) such that if $(a_0, d_0) \in V$, then (2) with $(a, d) = (a_0, d_0)$ has a unique positive solution.*

In the theorem above, the condition $0 < c, e < 1$, which biologically means the less competition rates of the two species, played an important role. Actually, in their proofs that implied the invertibility of Frechet derivative (linearization) of (2) at a fixed reproduction rate (a, a).

In the above model, the rate of change of densities largely depended on the constant (reproduction, self-limitation and competition rates) multiples of densities. However, in practice, the rate of change of densities may vary in a more complicated and irregular manner. Therefore, in the last decade, many research works have been focused on existence and uniqueness of steady state to the general competition model of two species of animals

$$\begin{cases} \left.\begin{array}{l} u_t(x,t) = \Delta u(x,t) + u(x,t)(g_1(u(x,t)) - g_2(v(x,t))) \\ \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(h_1(v(x,t)) - h_2(u(x,t))) \end{array}\right\} \text{ in } \Omega \times R^+, \\ u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0, \end{cases}$$

or equivalently, the positive solution to

$$\begin{cases} \left.\begin{array}{l} \Delta u(x) + u(x)(g_1(u(x)) - g_2(v(x))) = 0 \\ \\ \Delta v(x) + v(x)(h_1(v(x)) - h_2(u(x))) = 0 \end{array}\right\} \text{ in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where g_1, h_1 mean reproduction and self-limitation and g_2, h_2 imply competition (see [8], [9], [10], [11]). There are some existence results (see [10]), but few uniqueness result for the positive steady state solution of (3).

In [8], we established the following uniqueness result that generalizes Theorem 1.1.

Theorem 1.3. *Suppose:*

($U1$) $g_1, g_2, h_1, h_2 \in C^1$,

($U2$) $g_1, h_1, -g_2, -h_2$ *are strictly decreasing,*

($U3$) $g_2(0) = h_2(0) = 0$,

($U4$) *there are* $k_1, k_2 > 0$ *such that* $g_1(u) < 0$ *for* $u > k_1$ *and* $h_1(v) < 0$ *for* $v > k_2$,

($U5$) $g_1(0) - g_2(k_2) > \lambda_1, h_1(0) - h_2(k_1) > \lambda_1$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta$ *with homogeneous boundary condition,*

($U6$)

$$4 \inf_R(-g_1') \inf_R(-h_1') \geq \sup_\Omega\left(\frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}}\right)(\sup_R(g_2'))^2 + \sup_\Omega\left(\frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}\right)(\sup_R(h_2'))^2 + 2\sup_R(g_2')\sup_R(h_2').$$

Then (3) has a unique coexistence state.

Biologically, we can interpret the condition in Theorem 1.3 as follows. The functions g_1, g_2, h_1, h_2 describe how species 1 (u) and 2 (v) interact among themselves and with each other. Hence, the both conditions ($U5$) and ($U6$) imply that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1.

Actually, the significance of the global uniqueness is to investigate the stability of the positive steady state solution. There are several stability results for the model with constant rates (see [3], [4], [7], [11]), but the stability of the steady state solution for the general model still remains open.

The question in this paper concerns small perturbation of g_1, h_1 without losing the uniqueness of coexistence state of (3) when g_1, h_1 are nicer bounded functions, which will generalize Theorem 1.2. The conclusion says the two species may have small relaxation with which they can still coexist peacefully.

2. Preliminaries

In this section we will state some preliminary results which will be useful for our later arguments.

Definition 2.1. (Super and Sub Solutions) Consider the problem

$$\begin{cases} \Delta u + f(x,u) = 0 \ \text{ in } \ \Omega, \\ \\ u|_{\partial\Omega} = 0 \end{cases} \tag{4}$$

where $f \in C^{\alpha}(\bar{\Omega} \times R)$ and Ω is a bounded domain in R^n.

(A) A function $\bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \bar{u} + f(x,\bar{u}) \leq 0 \ \text{ in } \ \Omega, \\ \\ \bar{u}|_{\partial\Omega} \geq 0 \end{cases}$$

is called an super solution to (4).

(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x,\underline{u}) \geq 0 \ \text{ in } \ \Omega, \\ \\ \underline{u}|_{\partial\Omega} \leq 0 \end{cases}$$

is called a sub solution to (4).

Lemma 2.1. *Let $f(x,\xi) \in C^{\alpha}(\bar{\Omega} \times R)$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\bar{\Omega})$ be respectively, super and sub solutions to (4) which satisfy $\underline{u}(x) \leq \bar{u}(x), x \in \bar{\Omega}$. Then (4) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x) \leq u(x) \leq \bar{u}(x), x \in \bar{\Omega}$.*

We also need some information on the solutions of the following logistic equations.

Lemma 2.2. (see [10]) *Let*

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition. We denote this unique positive solution as θ_f.

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1$, the unique positive solution of

$$\begin{cases} \Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

is denoted by $\omega_a \equiv \theta_{a-x}$. Hence, θ_a is increasing as $a > 0$ is increasing.

3. Uniqueness with Perturbation

We consider the model

$$\begin{cases} \Delta u + u(g_1(u) - g_2(v)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ \Omega, \\ \Delta v + v(h_1(v) - h_2(u)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{5}$$

Here Ω is a bounded, smooth domain in R^n and:

$(P1)$ $g_1, h_1 \in C_B^{1,\alpha}$, $g_2, h_2 \in C^1$, where $C_B^{m,\alpha}$ in general, is the set of decreasing, bounded and continuous functions up to m-th order partial derivatives whose m-th order partial derivatives are Hölder continuous with exponent α.

$(P2)$ $g_1(0) > \lambda_1, h_1(0) > \lambda_1$, g_2, h_2 are strictly increasing and $g_2(0) = h_2(0) = 0$,

$(P3)$ there are $k_1, k_2 > 0$ such that $g_1(u) < 0$ for $u \geq k_1$ and $h_1(v) < 0$ for $v \geq k_2$.

The following is the main theorem.

Theorem 3.1. *Suppose:*

(A) $g_1(0) > \lambda_1(g_2(\theta_{h_1})), h_1(0) > \lambda_1(h_2(\theta_{g_1}))$, *where in general,* $\lambda_1(q)$ *is the smallest eigenvalue of* $-\Delta + q$ *with homogeneous boundary condition, denoted by simply* λ_1 *when* $q \equiv 0$.

(B) *(5) has a unique coexistence state* (u, v),

(C) *the Frechet derivative of (5) at* (u, v) *is invertible.*

Then there is a neighborhood V *of* (g_1, h_1) *in* $C_B^{1,\alpha}$ *such that if* $(g, h) \in V$, *then (5) with* $(g_1, h_1) = (g, h)$ *has a unique coexistence state.*

Theorem 3.1 looks like the consequence of Implicit Function Theorem. But the Inverse Function Theorem only guaranteed the uniqueness locally. Theorem 3.1 concluded the global uniqueness. The techniques we will use includes naturally Implicit Function Theorem and a priori estimates on solutions of (5).

Biologically, the first condition in this theorem indicates that the rates of self-reproduction is large. The condition of invertibility of Frechet derivative also illustrates that the rates of self-limitation is relatively larger than those of competitions which will be in Theorem 3.3. Then the conclusion says that small perturbation of reproduction and self-limitation rates does not affect to the existence and uniqueness of positive steady state, i.e. they can still coexist peacefully even if there is some slight change of reproduction and self-limitation rates.

Proof. Since the Frechet derivative of (5) at (u, v) is invertible, by the Implicit Function Theorem, there is a neighborhood V of (g_1, h_1) in $C_B^{1,\alpha}$ and a neighborhood W of (u, v) in $[C_0^{2,\alpha}(\bar{\Omega})]^2$ such that for all $(g, h) \in V$, there is a unique positive solution $(u_0, v_0) \in W$ of (5). Suppose the conclusion of the theorem is false. Then there are sequences $(\alpha_n, \beta_n, u_n, v_n), (\alpha_n, \beta_n, u_n^*, v_n^*)$ in $V \times [C_0^{2,\alpha}(\bar{\Omega})]^2$ such that (u_n, v_n) and

(u_n^*, v_n^*) are the positive solutions with $(g_1, h_1) = (\alpha_n, \beta_n)$ and $(u_n, v_n) \neq (u_n^*, v_n^*)$ and $(\alpha_n, \beta_n) \to (g_1, h_1)$. By the Schauder's estimate in elliptic theory and the solution estimate in the proof of Theorem 1.3, there is a constant $c > 0$ such that

$$|u_n|_{2,\alpha} \le c \sup_{x\in\bar{\Omega}}(u_n(x)) \le \sup_{x\in\bar{\Omega}}(\theta_{\alpha_n}(x)),$$

$$|v_n|_{2,\alpha} \le c \sup_{x\in\bar{\Omega}}(v_n(x)) \le \sup_{x\in\bar{\Omega}}(\theta_{\beta_n}(x)),$$

for all $n = 1, 2, \dots$.

But, by the convergence of $\{\alpha_n\}$, $\{\beta_n\}$ and the monotonicity of θ_f, we conclude that $|u_n|_{2,\alpha}$ and $|v_n|_{2,\alpha}$ are uniformly bounded. So, there are uniformly convergent subsequences of $\{u_n\}$ and $\{v_n\}$, again will be denoted by $\{u_n\}$ and $\{v_n\}$.

Let

$$(u_n, v_n) \to (\bar{u}, \bar{v}),$$

$$(u_n^*, v_n^*) \to (u^*, v^*).$$

Then $(\bar{u}, \bar{v}), (u^*, v^*) \in (C^{2,\alpha})^2$ are also solutions of (5). Claim $\bar{u} > 0, \bar{v} > 0, u^* > 0, v^* > 0$. It is enough to show that $\bar{u}$ and $\bar{v}$ are not identically zero because of the Maximum Principle. Suppose not. Then by the Maximum Principle again, one of the following cases should hold:

(1) $\bar{u}$ is identically zero and $\bar{v} > 0$.

(2) $\bar{u} > 0$ and $\bar{v}$ is identically zero.

(3) $\bar{u}$ is identically zero and $\bar{v}$ is identically zero.

Without loss of generality, assume $\bar{u}$ is identically zero.

Let $\tilde{u_n} = \frac{u_n}{\|u_n\|_\infty}, \tilde{v_n} = v_n$ for all $n \in N$. Then

$$\begin{cases} \Delta\tilde{u_n} + \tilde{u_n}(\alpha_n(u_n) - g_2(\tilde{v_n})) = 0 \\ \Delta\tilde{v_n} + \tilde{v_n}(\beta_n(\tilde{v_n}) - h_2(u_n)) = 0 \end{cases} \text{in } \Omega.$$

From the elliptic theory, $\tilde{u_n} \to \tilde{u}$ and

$$\begin{cases} \Delta\tilde{u} + \tilde{u}(g_1(0) - g_2(\bar{v})) = 0 \\ \Delta\bar{v} + \bar{v}h_1(\bar{v}) = 0 \end{cases} \text{in } \Omega,$$

by the continuity and uniform convergence, i.e., $g_1(0) = \lambda_1(g_2(\bar{v}))$.

(1) If $\bar{v} \equiv 0$, then by the monotonicity of g_2 and λ_1, $g_1(0) = \lambda_1(g_2(\bar{v})) = \lambda_1(g_2(0)) \leq \lambda_1(g_2(\theta_{h_1}))$ which contradicts our assumption.

(2) If $\bar{v}$ is not identically zero, then $\bar{v} = \theta_{h_1}$ and so $g_1(0) = \lambda_1(g_2(\bar{v})) = \lambda_1(g_2(\theta_{h_1}))$ which is also a contradiction to our assumption.

Consequently, $(\bar{u}, \bar{v})$ and (u^*, v^*) are coexistence states for (g_1, h_1). But, since the coexistence state with respect to (g_1, h_1) is unique, $(\bar{u}, \bar{v}) = (u^*, v^*) = (u, v)$. But, since $(u_n, v_n) \neq (u_n^*, v_n^*)$, it contradicts the Implicit Function Theorem. $\square$

The proof of the theorem also tells us that if one of the species becomes extinct, in other word, if one is excluded by others, then that means the reproduction rates are small, i.e. the region condition of reproduction rates (A) is reasonable.

Theorem 3.2. *If $(\alpha_n, \beta_n, u_n, v_n) \to (g_1, h_1, u, v)$ and if $u \equiv 0$ or $v \equiv 0$, then*

$$g_1(0) \leq \lambda_1(g_2(\theta_{h_1})) \text{ or } h_1(0) \leq \lambda_1(h_2(\theta_{g_1})).$$

The condition, invertibility of Frechet derivative, in Theorem 3.1 is too artificial. Now we turn out attention to get conditions to guarantee the invertibility of the Frechet derivative.

Theorem 3.3. *Suppose (u, v) is a positive solution to (5).*
If $4 \inf_R(-g_1') \inf_R(-h_1')uv > [(\sup_R g_2')u + (\sup_R h_2')v]^2$, then the Frechet derivative of (5) at (u, v) is invertible.

Proof. The solution operator for (5) is $A : (C^{2,\alpha})^2 \to (C^\alpha)^2$ such that for all $(v_1, v_2) \in (C^{2,\alpha})^2$,

$$A((v_1, v_2)) = (w_1, w_2),$$

where

$$w_1 = -\Delta v_1 - v_1(g_1(v_1) - g_2(v_2)),$$

$$w_2 = -\Delta v_2 - v_2(h_1(v_2) - h_2(v_1)).$$

The Frechet derivative at (u, v) is

$$A = \begin{pmatrix} -\Delta - (g_1(u) - g_2(v)) - ug_1'(u) & ug_2'(v) \\ vh_2'(u) & -\Delta - (h_1(v) - h_2(u)) - vh_1'(v) \end{pmatrix}.$$

We need to show that $N(A) = \{0\}$ by Fredholm alternative. If

$$\begin{cases} -\Delta\varphi - [(g_1(u) - g_2(v)) + ug_1'(u)]\varphi + g_2'(v)u\psi = 0, \\ -\Delta\psi + h_2'(u)v\varphi - [(h_1(v) - h_2(u)) + vh_1'(v)]\psi = 0, \end{cases}$$

then

$$\textstyle\int_\Omega [|\nabla\varphi|^2 - \{(g_1(u) - g_2(v)) + ug_1'(u)\}\varphi^2 + g_2'(v)u\varphi\psi] = 0,$$

$$\textstyle\int_\Omega [|\nabla\psi|^2 + h_2'(u)v\varphi\psi - \{(h_1(v) - h_2(u)) + vh_1'(v)\}\psi^2] = 0.$$

Since $\lambda_1(g_2(v) - g_1(u)) = \lambda_1(h_2(u) - h_1(v)) = 0$,

$$\textstyle\int_\Omega [|\nabla\varphi|^2 - (g_1(u) - g_2(v))\varphi^2] \geq 0,$$

$$\textstyle\int_\Omega [|\nabla\psi|^2 - (h_1(v) - h_2(u))\psi^2] \geq 0.$$

Hence,

$$\textstyle\int_\Omega (ug_1'(u)\varphi^2 + g_2'(v)u\varphi\psi) \leq 0,$$

$$\textstyle\int_\Omega (h_2'(u)v\varphi\psi + h_1'(v)v\psi^2) \leq 0.$$

Hence, $\int_\Omega [ug_1'(u)\varphi^2 + (g_2'(v)u + h_2'(u)v)\varphi\psi + h_1'(v)v\psi^2] \leq 0$.

Hence, if $4\inf_R(-g_1')\inf_R(-h_1')uv > [(\sup_R g_2')u + (\sup_R h_2')v]^2$, then the integrand in the left side is positive definite form in Ω, which means $\varphi \equiv \psi \equiv 0$. Therefore, the above Frechet derivative A is invertible. $\square$

Combining Theorem 1.3, Theorem 3.1 and Theorem 3.3, we have the following corollary, which is actually the main result in this section.

Corollary 3.4. *Suppose:*

(A) $g_1(0) > \lambda_1 + g_2(k_2)$, $h_1(0) > \lambda_1 + h_2(k_1)$, *and*

(B)

$$4\inf_R(-g_1')\inf_R(-h_1') > [\sup_R(g_2')+\sup_R(h_2')\sup_\Omega \frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}][\sup_R(g_2')\sup_\Omega \frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}}+\sup_R(h_2')].$$

Then there is a neighborhood V of (g_1,h_1) in C_B^1 such that if $(g,h)\in V$, then (5) with $(g_1,h_1)=(g,h)$ has a unique coexistence state.

Proof. From $\theta_{g_1}<k_1,\ \theta_{h_1}<k_2$ and the monotonicity of g_2,h_2 we have

$$\begin{cases} g_1(0)>\lambda_1+g_2(k_2)\geq\lambda_1(g_2(\theta_{h_1})),\\ h_1(0)>\lambda_1+h_2(k_1)\geq\lambda_1(h_2(\theta_{g_1})).\end{cases}$$

$$\begin{aligned}
&4\inf_R(-g_1')\inf_R(-h_1')\\
>\ &[\sup_R(g_2')+\sup_R(h_2')\sup_\Omega\frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}][\sup_R(g_2')\sup_\Omega\frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}}\\
&+\sup_R(h_2')]\\
=\ &[\sup_R(g_2')]^2\sup_\Omega\frac{\theta_{g_1}}{\theta_{h_1-h_2(k_2)}}+\sup_R(g_2')\sup_R(h_2')\\
&+\sup_\Omega\frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}}\sup_\Omega\frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}\sup_R(g_2')\sup_R(h_2')\\
&+[\sup_R(h_2')]^2\sup_\Omega\frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}\\
\geq\ &[\sup_R(g_2')]^2\sup_\Omega\frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}}+2\sup_R(g_2')\sup_R(h_2')\\
&+[\sup_R(h_2')]^2\sup_\Omega\frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}
\end{aligned}$$

since $\theta_{g_1} > \theta_{g_1 - g_2(k_2)}, \theta_{h_1} > \theta_{h_1 - h_2(k_2)}$.

Therefore, (5) has a unique coexistence state (u, v) from Theorem 1.3. Furthermore, by estimate of the solution in the proof of Theorem 1.3,

$$\begin{aligned} & 4 \inf_R(-g_1') \inf_R(-h_1') \\ > \; & [\sup_R(g_2') + \sup_R(h_2') \sup_\Omega \frac{\theta_{h_1}}{\theta_{g_1-g_2(k_2)}}][\sup_R(g_2') \sup_\Omega \frac{\theta_{g_1}}{\theta_{h_1-h_2(k_1)}} \\ & + \sup_R(h_2')] \\ \geq \; & [\sup_R(g_2') + \sup_R(h_2') \frac{v}{u}][\sup_R(g_2') \frac{u}{v} + \sup_R(h_2')]. \end{aligned}$$

Thus, we obtain

$$4 \inf_R(-g_1') \inf_R(-h_1') uv > [\sup_R(g_2') u + \sup_R(h_2') v]^2.$$

It implies that the Frechet derivative of (5) at (u, v) is invertible from Theorem 3.2. Therefore, the result follows from Theorem 3.1. $\square$

4. Amount of Perturbation

In this section, we discuss how large we can perturb g_1 and h_1 without losing the uniqueness of the positive solution. We consider the model with the perturbation of (g_1, h_1)

$$\begin{cases} \Delta u + u(\sigma g_1(u) - g_2(v)) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(\sigma h_1(v) - h_2(u)) = 0 & \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, & \end{cases} \tag{6}$$

where Ω is a bounded domain in R^n, $g_2, h_2 \in C^1$ are strictly increasing functions with $g_2(0) = h_2(0) = 0$ and $1 \leq \sigma \leq c$ is a parameter.

Theorem 4.1. *Suppose:*

(A) (6) has a unique positive solution for $g_1, h_1 \in C_B^{1,\alpha}$ $(\sigma = 1)$,

(B) for all

$$1 \leq \sigma \leq c, \quad \sigma g_1(0) > \lambda_1(g_2(\theta_{\sigma h_1})), \quad \sigma h_1(0) > \lambda_1(h_2(\theta_{\sigma g_1})),$$
$$g_1(u) < 0, \quad h_1(v) < 0$$

for $u, v \geq k > 0$, the Frechet derivative of (6) with $(\sigma g_1, \sigma h_1)$ at every positive solution to (6) is invertible.

Then (6) has a unique positive solution with $(\sigma g_1, \sigma h_1)$ for all $1 \leq \sigma \leq c$. Furthermore, there is an open set W in $(C_B^{1,\alpha})^2$ such that $\{(\sigma g_1, \sigma h_1) : 1 \leq \sigma \leq c\} \subseteq W$ and for every $(g, h) \in W$, (6) with (g, h) has a unique positive solution.

Proof. Since (6) with (g_1, h_1) has a unique positive solution (u, v) and the Frechet derivative of (6) at (u, v) is invertible, Theorem 3.1 implies that there is $1 < \sigma' \leq c$ such that if $1 \leq \sigma \leq \sigma'$, then (6) has a unique positive solution.

Let

$$\lambda_s = \sup\{1 < \sigma' \leq c : \text{(6) has a unique coexistence state for all } 1 \leq \sigma \leq \sigma'\}.$$

We need to show that $\lambda_s = c$. Suppose $\lambda_s < c$. From the definition of λ_s, there are sequences $\{\sigma_n\}$ and $\{(u_n, v_n)$ such that $\sigma_n \to \lambda_s^-$ and (u_n, v_n) is the unique positive solution of (6) with $(\sigma_n g_1, \sigma_n h_1) \to (\lambda_s g_1, \lambda_s h_1)$. Then by the elliptic theory as we mentioned in Theorem 3.1, there is (u, v) such that (u_n, v_n) converges to (u, v) uniformly and (u, v) is a solution to (6) with $(\lambda_s g_1, \lambda_s h_1)$. We claim that u and v are not identically zero. Suppose this is false. Without loss of generality, assume u is identically zero.

Let $\widetilde{u_n} = \dfrac{u_n}{\| u_n \|_\infty}$ for all $n \in N$. Then

$$\Delta \widetilde{u_n} + \widetilde{u_n}(\sigma_n g_1(u_n) - g_2(v_n)) = 0,$$

$$\Delta v_n + v_n(\sigma_n h_1(v_n) - h_2(u_n)) = 0,$$

and $\widetilde{u_n} \to \widetilde{u}$ uniformly in Ω by elliptic theory again, and

$$\Delta\widetilde{u} + \widetilde{u}(\lambda_s g_1(0) - g_2(v)) = 0,$$

$$\Delta v + v\lambda_s h_1(v) = 0.$$

It implies that $\lambda_s g_1(0) = \lambda_1(g_2(v))$.

(1) If v is identically zero, then by the monotonicity of g_2 and λ_1, $\lambda_s g_1(0) = \lambda_1(g_2(0)) < \lambda_1(g_2(\theta_{\lambda_s h_1}))$ which is a contradiction to our assumption.

(2) If v is not identically zero, then $v = \theta_{\lambda_s h_1}$ and so $\lambda_s g_1(0) = \lambda_1(g_2(\theta_{\lambda_s h_1}))$ which is also a contradiction to our assumption.

Thus u and v are not identically zero. We claim that (6) with $(\lambda_s g_1, \lambda_s h_1)$ has a unique coexistence state. In fact, if not, assume that $(\bar{u}, \bar{v}) \neq (u, v)$ is another coexistence state. By Implicit Function Theorem, there exists $1 < \bar{\sigma} < \lambda_s$ which is very close to λ_s and (6) with $(\bar{\sigma} g_1, \bar{\sigma} h_1)$ has a coexistence state very close to $(\bar{u}, \bar{v})$ which means that (6) with $(\bar{\sigma} g_1, \bar{\sigma} h_1)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (6) with $(\lambda_s g_1, \lambda_s h_1)$ has a unique coexistence state and a invertible Frechet derivative, Theorem 3.1 concluded that λ_s cannot be as defined. Therefore, for each $1 \le \sigma \le c$, (6) with $(\sigma g_1, \sigma h_1)$ has a unique coexistence state. Furthermore, by the assumption, for each $1 \le \sigma \le c$, the Frechet derivative of (6) with $(\sigma g_1, \sigma h_1)$ at the unique solution is invertible. Hence, Theorem 3.1 concluded that there is an open set W in $(C_B^{1,\alpha})^2$ satisfying the theorem as stated. $\square$

Corollary 4.2. *Suppose:*

(A) there exist $k_1, k_2 > 0$ such that for all $1 \le \sigma \le c$, $\sigma g_1(0) > \lambda_1 + g_2(k_2), \sigma h_1(0) > \lambda_1 + h_2(k_1)$ and $g_1(u) < 0, h_1(v) < 0$ for $u \ge k_1, v \ge k_2$,

(B) The following inequalities is valid

$$4\inf_R(-g_1')\inf_R(-h_1') > [\sup_R(g_2') + \sup_R(h_2') \sup_{1\le\sigma\le c}\sup_\Omega \frac{\theta_{\sigma h_1}}{\theta_{\sigma g_1 - g_2(k_2)}}] \times [\sup_R(g_2') \sup_{1\le\sigma\le c}\sup_\Omega \frac{\theta_{\sigma g_1}}{\theta_{\sigma h_1 - h_2(k_1)}} + \sup_R(h_2')].$$

Then there is an open set W in $(C_B^{1,\alpha})^2$ such that $\{(\sigma g_1, \sigma h_1) : 1 \leq \sigma \leq c\} \subseteq W$ and for every $(g,h) \in W$, (6) with (g,h) has a unique positive solution.

Proof. From $\theta_{\sigma g_1} < k_1, \theta_{\sigma h_1} < k_2$ and the monotonicity of g_2, h_2, we have

$$\sigma g_1(0) > \lambda_1 + g_2(k_2) \geq \lambda_1(g_2(\theta_{\sigma h_1})),$$

$$\sigma h_1(0) > \lambda_1 + h_2(k_1) \geq \lambda_1(h_2(\theta_{\sigma g_1}))$$

for all $1 \leq \sigma \leq c$.

The condition (B) already guarantees $(U6)$ of Theorem 1.3, and so by Theorem 1.3, (6) with (g_1, h_1) has a unique positive solution. Furthermore, by the estimate of the solution in the proof of the Theorem 1.3, if (u,v) is a positive solution of (6) with $(\sigma g_1, \sigma h_1)$ for $1 \leq \sigma \leq c$, then

$$\begin{aligned}
& 4 \inf_R(-\sigma g_1') \inf_R(-\sigma h_1') uv \\
> \ & [\sigma u \sup_R(g_2') + \sigma u \sup_R(h_2') \sup_{1\leq\sigma\leq c} \sup_\Omega \tfrac{\theta_{\sigma h_1}}{\theta_{\sigma g_1 - g_2(k_2)}}] \\
& [\sigma v \sup_R(g_2') \sup_{1\leq\sigma\leq c} \sup_\Omega \tfrac{\theta_{\sigma g_1}}{\theta_{\sigma h_1 - h_2(k_1)}} + \sigma v \sup_R(h_2')] \\
\geq \ & [\sigma u \sup_R(g_2') + \sigma u \sup_R(h_2') \tfrac{v}{u}][\sigma v \sup_R(g_2') \tfrac{u}{v} + \sigma v \sup_R(h_2')] \\
= \ & [\sup_R(\sigma g_2') u + \sup_R(\sigma h_2') v]^2.
\end{aligned}$$

Hence, by the Theorem 3.3, if (u,v) is a positive solution of (6) with $(\sigma g_1, \sigma h_1)$ for $1 \leq \sigma \leq c$, then the Frechet derivative at (u,v) is invertible. Therefore, the corollary follows from Theorem 4.1. □

Next, we look at the stationary solutions denoting the existence of only one species in the same parameter regions($1 \leq \sigma \leq c$). By Lemma 2.2, for each $1 \leq \sigma \leq c$,

$$\begin{cases} \Delta u + u\sigma g_1(u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0 \end{cases}$$

and

$$\begin{cases} \Delta v + v\sigma h_1(v) = 0 \ \text{ in } \ \Omega, \\ v|_{\partial\Omega} = 0, v > 0 \end{cases}$$

have a unique positive solution $\theta_{\sigma g_1}$ and $\theta_{\sigma h_1}$, respectively. But, since $g_2(0) = h_2(0) = 0$, for each $1 \le \sigma \le c$, $(\theta_{\sigma g_1}, 0)$ and $(0, \theta_{\sigma h_1})$ are solutions to (6), which means either one of the two species may be extinct, in other words, one may be excluded by the other. But, by the comment after Lemma 2.2, the solutions are getting larger as σ is getting larger, in other words, it has the maximum solution when $\sigma = c$, which biologically implies that the existing species has larger density when the self-reproduction and self-limitation rates are larger.

5. The Case with Constant Growth Rates

In this section, we look at the uniqueness results of the case with constant growth rates that can be easily derived from the previous sections.

First of all, by Corollary 3.4, we have the following uniqueness result.

Corollary 5.1. *Suppose*

(A) $a > \lambda_1 + \frac{cd}{f}, d > \lambda_1 + \frac{ae}{b}$, *and*

(B) $4bf > (c + e \sup_\Omega \frac{\frac{1}{f}\omega_d}{\frac{1}{b}\omega_{a-\frac{cd}{f}}})(c \sup_\Omega \frac{\frac{1}{b}\omega_a}{\frac{1}{f}\omega_{d-\frac{ae}{b}}} + e)$.

Then there is a neighborhood V *of* (a, d) *in* R^2 *such that if* $(a_0, d_0) \in V$, *then (2) with* (a_0, d_0) *has a unique coexistence state.*

We also consider the model

$$\begin{cases} \Delta u + u(\sigma(a - bu) - cv) = 0 & \\ & \text{in } \Omega, \\ \Delta v + v(\sigma(d - fv) - eu) = 0 & \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, & \end{cases} \tag{7}$$

where $1 \le \sigma \le c$ is a parameter.

Then by Corollary 4.2, we have the following uniqueness result.

Corollary 5.2. *Suppose*

(A) *for all* $1 \leq \sigma \leq c$, $\sigma a > \lambda_1 + \frac{cd}{f}, \sigma d > \lambda_1 + \frac{ae}{b}$ *and*

(B) $4bf > (c + e \sup_{1\leq\sigma\leq c} \sup_{\Omega} \frac{\frac{1}{\sigma f}\omega_{\sigma d}}{\frac{1}{b}\omega_{\sigma a - \frac{cd}{f}}})(c \sup_{1\leq\sigma\leq c} \sup_{\Omega} \frac{\frac{1}{\sigma b}\omega_{\sigma a}}{\frac{1}{f}\omega_{\sigma d - \frac{ae}{b}}} + e)$.

Then there is an open set W *in* R *such that* $[1, c] \subseteq W$ *and for every* $\gamma \in W$, *(7) with* γ *in place of* σ *has a unique positive solution.*

It is important to note that the conditions in both Corollary 5.1 and Corollary 5.2 imply that the reproduction and self-limitation rates are relatively larger than competition rates.

References

[1] S.W. Ali, C. Cosner, On the uniqueness of the positive steady state for Lotka-Volterra Models with diffusion, *Journal of Mathematical Analysis and Application*, **168** (1992), 329-341.

[2] R.S. Cantrell, C. Cosner, On the steady-state problem for the Volterra-Lotka competition model with diffusion, *Houston Journal of Mathematics*, **13** (1987), 337-352.

[3] R.S. Cantrell, C. Cosner, On the uniqueness and stability of positive solutions in the Volterra-Lotka competition model with diffusion, *Houston J. Math.*, **15** (1989), 341-361.

[4] C. Cosner, A.C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *Siam J. Appl. Math.*, **44** (1984), 1112-1132.

[5] D. Dunninger, *Lecture Note for Applied Analysis in Michigan State University.*

[6] J.L. Gomez, R. Pardo, Existence and uniqueness for some competition models with diffusion, *C.R. Acad. Sci. Paris, Ser.I Math.*, **313** (1991), 933-938.

[7] C. Gui, Y. Lou, Uniqueness and nonuniqueness of coexistence states in the Lotka-Volterra competition model, *Comm. Pure and Appl. Math.*, **XVL2** , No. 12 (1994), 1571-1594.

[8] J. Kang, Y. Oh, A sufficient condition for the uniqueness of positive steady state to a reaction diffusion system, *Journal of Korean Mathematical Society*, **39**, No. 39 (2002), 377-385.

[9] J. Kang, Y. Oh, Uniqueness of coexistence state of general competition model for several competing species, *Kyungpook Mathematical Journal*, **42**, No. 2 (2002), 391-398.

[10] L. Li, R. Logan, Positive solutions to general elliptic competition models, *Differential and Integral Equations*, **4** (1991), 817-834.

[11] A. Leung, Equilibria and stabilities for competing-species, reaction-diffusion equations with Dirichlet boundary data, *J. Math. Anal. Appl.*, **73** (1980), 204-218