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LAMÉ EQUATION IN THE WEIERSTRASS FORM AND EXACT SOLVABILITY

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Abstract: In this paper, we construct some exactly solvable potentials giving rise to solutions of the Lamé equation in the Weierstrass form which can be written in terms of associated Legendre, Hermite, and associated Laguerre polynomials.

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1. Introduction

In what follows $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all integers, real numbers and complex numbers, respectively. Let $\mathcal{L} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subset \mathbb{C}$ be a lattice, with $\omega_j, j = 1, 2$ independent over $\mathbb{R}$, and let $\wp : \mathbb{C} \longrightarrow \mathbb{P}^1(\mathbb{C})$ be the corresponding Weierstrass $\wp-$function. Here, ω_1 and ω_2 are complex numbers usually called half-periods of the $\wp-$function $\wp(z)$ and $\mathbb{P}^1(\mathbb{C})$ is the complex projective space. The Lamé equation in the Weierstrass form, which we write in the self-adjoint form, is

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + (n(n+1)\wp(z) - E)\,\psi = 0, \tag{1.1}$$

where E and n are real or complex parameters. While n is called the degree parameter, E is the accessory (or perhaps, energy) parameter and, in many

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applications, it plays the role of eigenvalue. The equation (1.1) may be regarded as a differential equation defined on the complex torus $\mathbf{T} := \mathbb{C}/\mathcal{L}$ and accordingly all points of $\mathbb{C}$ may be reqarded as being expressed $\mod \mathcal{L}$. Now, let $\mathbf{I} \subset \mathbb{C}$ be an open, possibly an infinite interval. In particular, in the sequel, we may assume $\mathbf{I}$ is a subset of the real line $\mathbb{R}$ with $z \in \mathbf{I}$. In other words, $\mathbf{I}$ may be assumed to be the domain of variation of z.

The search for exactly solvable models of physical problems has become a topic of research in its own right. This is partly because a very small number of exactly solvable models are known up to date and partly because exactly solvable potentials represent useful aids in modelling realistic physical problems. Exactly solvable here means that the eigenvalues and the eigenfunctions of the models can be obtained analytically in closed form. Owing to their major interest (see Hounkonnou *et al* [3]), many methods have been developed in order to increase the number of exactly solvable potentials. Very prominent among the methods is the technique of operator factorization first introduced by Schrödinger ([10], [11], [12]), later developed by Infeld and Hall [4], and renewed by Witten [15] and Gendenshtein [2] in connection with supersymmetry quantum mechanics (SUSY QM) and the concept of shape invariance, respectively. Also in relation to SUSY QM, Levai ([5], [6], [7], [8]) has developed an elegant method of constructing solvable potentials for which Schrödinger equation can be solved exactly in terms of known special functions. This method, called the technique of *point canonical transformation*, is based on mapping the Schrödinger equation into the second order differential equation of a special function by appropriate variable transformations. In this paper, we use similar method to construct some exactly solvable potentials from the Lamé equation in the Weierstrass form giving rise to polynomial solutions of the equation.

The paper is arranged as follows. In Section 2, we set down the formalism for the construction of new exactly solvable potentials from known ones. The main result of the paper are presented in Section 3 where the formalism that has been set down in Section 2 is applied to construct three exactly solvable potentials which give rise to solutions of the Lamé equation which are expressed in terms of special orthogonal polynomials.

2. Construction of New Solvable Potentials

This section presents the technique for constructing new solvable potentials from known Schröedinger operators. Consider a Schröedinger operator equation

$$H\psi(z) \equiv \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right)\psi(z) = E\psi(z), \tag{2.1}$$

where $V(z)$ is the potential function and E is the energy eigenvalues. We assume that equation (2.1) has a solution of the form

$$\psi(z) = f(z)F(g(z)), \tag{2.2}$$

where $f(z)$ and $g(z)$ are differentiable functions of the complex variable z to be determined, and $F(g(z))$ is a special function satisfying a second order ordinary differential equation

$$\ddot{F}(g) + Q(g)\dot{F}(g) + R(g)F(g) = 0. \tag{2.3}$$

Here, $^{\bullet} =: \dfrac{\mathrm{d}}{\mathrm{d}g}$. The logarithmic derivative of the function in (2.2) is given by

$$\frac{\psi'(z)}{\psi(z)} = \frac{f'(z)}{f(z)} + \frac{\dot{F}(g(z))g'(z)}{F(g(z))}, \qquad ' =: \frac{\mathrm{d}}{\mathrm{d}z}. \tag{2.4}$$

By differentiating through equation (2.4), we obtain

$$\begin{aligned}\frac{\psi''(z)}{\psi(z)} - \left(\frac{f'(z)}{f(z)}\right)^2 - 2\frac{f'(z)}{f(z)}\frac{\dot{F}(g(z))g'(z)}{F(g(z))} - \left(\frac{\dot{F}(g(z))g'(z)}{F(g(z))}\right)^2 \\ = \frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2 + \frac{F(g(z))\ddot{F}(g(z))(g'(z))^2}{F(g(z))} \\ + \frac{\dot{F}(g(z))g''(z)}{F(g(z))} - \frac{(\dot{F}(g(z))g'(z))^2}{F(g(z))^2},\end{aligned}$$

which may be re-written as

$$\frac{\psi''(z)}{\psi(z)} = 2\frac{f'(z)}{f(z)}\frac{\dot{F}(g(z))g'(z)}{F(g(z))} + \frac{f''(z)}{f(z)} + \frac{\ddot{F}(g)(g'(z))^2}{F(g(z))} + \frac{\dot{F}(g(z))}{F(g(z))}g''(z). \tag{2.5}$$

By re-arranging equation (2.5), we obtain the differential equation (2.6) below

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+\left(\frac{\ddot{F}(g(z))(g'(z))^2}{F(g(z))}+\left(g''(z)+2\frac{f'(z)}{f(z)}g'(z)\right)\frac{\dot{F}(g(z))}{F(g(z))}+\frac{f''(z)}{f(z)}\right)\psi(z) = 0. \quad (2.6)$$

Equation (2.6) compared with equation (2.1) gives

$$V(z)-E=\frac{\ddot{F}(g(z))(g'(z))^2}{F(g(z))}+\left(g''(z)+2\frac{f'(z)}{f(z)}g'(z)\right)\frac{\dot{F}(g(z))}{F(g(z))}+\frac{f''(z)}{f(z)}$$

which may be re-written as

$$\ddot{F}(g(z))+\left(\frac{g''(z)}{(g'(z))^2}+2\frac{f'(z)}{f(z)}\frac{1}{g'(z)}\right)\dot{F}(g(z)) + \left(\frac{f''(z)}{f(z)}\frac{1}{(g'(z))^2}+\frac{E-V(z)}{(g'(z))^2}\right)F(g(z))=0 \quad (2.7)$$

The equation (2.7) compared with equation (2.3) gives

$$Q(g) = \frac{g''(z)}{(g'(z))^2}+2\frac{f'(z)}{f(z)}\frac{1}{g'(z)} \quad (2.8)$$

$$R(g) = \frac{f''(z)}{f(z)}\frac{1}{(g'(z))^2}+\frac{E-V(z)}{(g'(z))^2} \quad (2.9)$$

It follows from equation (2.8) that

$$\frac{f'(z)}{f(z)}=-\frac{g''(z)}{2g'(z)}+\frac{g'(z)}{2}Q(g(z)) \quad (2.10)$$

Now, by solving equation (2.10) above we get

$$f(z)=(g'(z))^{-\frac{1}{2}}\exp\left(\frac{1}{2}\int^{g(z)}Q(u)\mathrm{d}u\right) \quad (2.11)$$

Note that $f(z)$ plays the role of ground-state eigenfunction. To obtain $R(g)$ in terms of g and its derivatives only, from equation (2.10) we consider

$$\frac{f''(z)}{f(z)}-\left(\frac{f'(z)}{f(z)}\right)^2 = -\frac{1}{2}\frac{g'(z)g'''(z)-(g''(z))^2}{(g'(z))^2} +\frac{1}{2}\left(g''(z)Q(g)-(g'(z))^2\dot{Q}(g(z))\right).$$

Coming from (2.10) the last equation becomes

$$\frac{f''(z)}{f(z)} = \left(\frac{g''(z)}{2g'} + \frac{g'(z)}{2}Q(g)\right)^2 - \frac{1}{2}\frac{g'(z)g'''(z) - (g''(z))^2}{(g'(z))^2} + \frac{1}{2}\left(g''(z)Q(g(z)) - (g'(z))^2\dot{Q}(g(z))\right). \quad (2.12)$$

By using equation (2.12), we can now re-write equation (2.9) as

$$R(g) = \frac{1}{(g'(z))^2}\left[\frac{3}{4}\left(\frac{g''(z)}{g'(z)}\right)^2 + g''(z)Q(g) - \frac{1}{2}\frac{g'''(z)}{g'(z)} + \frac{(g'(z))^2(Q(g))^2}{4} - \frac{(g'(z))^2}{2}\dot{Q}(g) + \frac{E - V(z)}{(g'(z))^2}\right] \quad (2.13)$$

Equation (2.13) is re-arranged to obtain

$$\begin{aligned} E - V(z) &= \frac{1}{2}\frac{g'''(z)}{g'(z)} - \frac{3}{4}\left(\frac{g''(z)}{g'(z)}\right)^2 \\ &\quad - g''(z)Q(g) + (g'(z))^2\left(R(g) - \frac{1}{2}\dot{Q}(g) - \frac{1}{4}(Q(g))^2\right). \end{aligned} \quad (2.14)$$

In the next section, we apply this formalism to associated Legendre, Hermite and associated Laguerre orthogonal polynomials and find new exactly solvable potentials relative to the Lamé equation in the Weierstrass form.

3. Main Results

In this section, we apply the formalism explained in Section 2 to (orthogonal polynomials as) special functions. We start by applying the method to associated Legendre polynomials in Subsection 3.1. We continue in Subsection 3.2 by applying the formalism to Hermite polynomials. The section ends after applying the formalism to associated Laguerre polynomials in Subsection 3.3.

3.1. Exactly Solvable Potentials for Associated Legendre Polynomials

In this subsection, we apply the procedure described in Section 2 to associated Legendre polynomials. The differential equation of the associated Legendre polynomials(cf:Whittaker and Watson [14], p.205) may be written as

$$\ddot{F}(g) - \frac{2g}{1-g^2}\dot{F}(g) + \left(\frac{\nu(\nu+1)}{1-g^2} - \frac{\mu^2}{(1-g^2)^2}\right) F(g) = 0, \tag{3.1}$$

where, μ and ν are positive integers. This equation which plays the role of equation (2.3) is satisfied by

$$P_\nu^\mu(g(z)) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+g(z)}{1-g(z)}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-g(z)}{2}\right)$$

(see Wang *et al* [13], p. 249, §15.6(8)) where, $|\arg(g(z)\pm 1| < 1$, Γ is the gamma function and ${}_2F_1$ is the hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; g) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+\alpha)n!} g^n.$$

We can see, by comparing equation (3.1) with equation (2.3), that

$$R(g) = \left(\frac{\nu(\nu+1)}{1-g^2} - \frac{\mu^2}{(1-g^2)^2}\right) \quad \text{and} \quad Q(g) = -\frac{2g}{1-g^2}.$$

Substituting $Q(g)$ into equation (2.11) yields

$$\begin{aligned} f(z) &= (g'(z))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{g(z)} -\frac{2u}{1-u^2}\mathrm{d}u\right) \\ &= (g'(z))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\log_e(1-g^2)\right) \\ &= (g'(z))^{-\frac{1}{2}}(1-g^2)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

Also, substituting the expressions for $Q(g)$ and $R(g)$ into (2.14) with $E = \wp(\varepsilon)$ and $V(z) = 2\wp(z)$, we obtain

$$\begin{aligned} \wp(\varepsilon) - 2\wp(z) &= \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - g''\left(-\frac{2g}{1-g^2}\right) \\ &\quad + (g')^2\left[\left(\frac{\nu(\nu+1)}{1-g^2} - \frac{\nu^2}{(1-g^2)^2}\right)\right. \\ &\qquad \left. -\frac{1}{2}\left\{-\frac{2(1-g^2)+(2g)^2}{(1-g^2)^2}\right\} - \frac{1}{4}\left(\frac{-2g}{1-g^2}\right)^2\right] \end{aligned}$$

$$= \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \left(\frac{2gg''}{1-g^2}\right) + [\nu(\nu+1)-\mu^2]\frac{(g')^2}{(1-g^2)^2} + \frac{(g^2+1)(g')^2}{(1-g^2)^2} + \frac{g^2(g')^2}{(1-g^2)^2}. \tag{3.3}$$

Since we have to get a constant ($E = \wp(\varepsilon)$) on the left-hand side of equation (3.3), there must be at least one term on the right-hand side, from which a constant arises. Thus, we assume that $g(z)$ satisfies the differential equation $(g')^2(1-g^2)^{-1} = a$ (constant). This equation is solved, for instance, by $g(z) = \sin az$. Therefore,

$$g'(z) = a\cos az, g''(z) = -a^2 \sin az = -a^2 g, g'''(z) = -a^3 \cos az = a^2 g'(z).$$

Substituting these into equation (3.3), we obtain

$$\begin{aligned} \wp(\varepsilon) - 2\wp(z) &= \frac{a^2}{2} - \frac{3g^2}{4(1-g^2)^2} - \frac{2a^2g^2}{1-g^2} + \nu(\nu+1)a^2 \\ &\quad - \frac{\mu^2 a^2}{1-g^2} + \frac{(1+g)a^2}{1-g^2} + \frac{a^2g^2}{1-g^2} \\ &= [\nu(\nu+1) - \frac{1}{2}]a^2 \\ &\quad + \frac{1}{1-g^2}\left[\frac{3g^2}{1-g^2} - 2a^2g^2 - \mu^2 + (1+g)a^2 + a^2g^2\right]. \end{aligned} \tag{3.4}$$

We can now deduce an explicit expression for $E = \wp(\varepsilon)$ by equating it to the constant part of the right-hand side of equation (3.4) so that

$$\begin{aligned} [\nu(\nu+1) - \frac{1}{2}]a^2 &= \wp(\varepsilon). \\ \therefore\ a &= \pm\sqrt{\frac{2\wp(\varepsilon)}{2\nu(\nu+1)-1}}\ . \end{aligned} \tag{3.5}$$

Thus,

$$g(z) = \sin\left(\pm z\sqrt{\frac{2\wp(\varepsilon)}{2\nu(\nu+1)-1}}\right),$$

and

$$f(z) = (\sqrt{a})^{-1}\sqrt{\cos az} = \sqrt{\pm\sqrt{\frac{2\nu(\nu+1)-1}{2\wp(\varepsilon)}}}\sqrt{\cos\left(\pm z\sqrt{\frac{2\wp(\varepsilon)}{2\nu(\nu+1)-1}}\right)}.$$

Therefore, the function $\psi(z)$ of equation (2.2), in this case, takes the form

$$\begin{aligned}\psi(z) \equiv \psi_\nu^\mu(z) &= (\sqrt{a})^{-1}\sqrt{\cos az}\ \ P_\nu^\mu(g)\\ &= \sqrt{\pm\sqrt{\frac{2\nu(\nu+1)-1}{2\wp(\varepsilon)}}}\sqrt{\cos\left(\pm z\sqrt{\frac{2\wp(\varepsilon)}{2\nu(\nu+1)-1}}\right)}\\ &\quad \times P_\nu^\mu\left(\sin\left(\pm z\sqrt{\frac{2\wp(\varepsilon)}{2\nu(\nu+1)-1}}\right)\right).\end{aligned}$$

The new exactly solvable potential is obtained from equation (3.4) as

$$\begin{aligned}V(z) &= \frac{1}{1-g^2}\left[\frac{3g^2}{1-g^2} - 2a^2g^2 - \mu^2 + (1+g)a^2 + a^2g^2\right]\\ &= [3\tan^2\ az - \mu^2 + \sin\ az]\sec^2\ az - a^2\tan^2\ az.\end{aligned}$$

3.2. Exactly Solvable Potentials for Hermite Polynomials

In this subsection, the formalism of Section 2 is applied to Hermite polynomials. We know (Erdélyi[1], §10.31, p.193) that the differential equation of the Hermite polynomials

$$H_\nu(g) = (-1)^\nu e^{g^2}\frac{\mathrm{d}^\nu}{\mathrm{d}g^\nu}(e^{-g^2})$$

with $g := g(z)$ is given by

$$\ddot{F}(g(z)) - 2g\dot{F}(g(z)) + 2\nu F(g(z)) = 0.$$

Then, the coefficients are $Q(g) = -2g$ and $R(g) = 2\nu$, and

$$\begin{aligned}f(z) &= (g')^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\int^g(-2u)\mathrm{d}u\right)\\ &= (g')^{-\frac{1}{2}}\exp(\frac{1}{2}g^2).\end{aligned} \tag{3.6}$$

The current form of equation (2.14) becomes

$$\begin{aligned}\wp(\varepsilon) - 2\wp(z) &= \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - g''Q(g) + (g')^2\left(R(g)\right.\\ &\quad \left.-\frac{1}{2}\dot{Q}(g) - \frac{1}{4}Q(g)^2\right)\end{aligned}$$

$$= \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - 2g.g'' + (g')^2\left(2\nu + 1 - g^2\right) \quad (3.7)$$

One term at least on the right-hand side of (3.7) must be constant to play the role of $E := \wp(\varepsilon)$, energy term of the left-hand side. This condition is achieved if we let $(g')^2 g^2 = b(\text{constant})$ or $(g')^2 = \text{constant}$ [but $(g')^2 = \text{constant}$ is linear] . A possible solution of the equation $(g')^2 g^2 = b(\text{constant})$ which we consider here for our analysis is $g(z) = \left(2z\sqrt{b}\right)^{\frac{1}{2}}$ so that

$$g'(z) = b^{\frac{1}{2}} g(z)^{-1}, \quad \frac{g''(z)}{g'(z)} = -b^{\frac{1}{2}} g(z)^{-2}, \quad \frac{g'''(z)}{g'(z)} = -3bg(z)^{-4}.$$

Substituting these expressions into equation (3.7), we obtain

$$\begin{aligned} \wp(\varepsilon) - 2\wp(z) &= \frac{3}{4}bg(z)^{-4} - \frac{3}{4}bg(z)^{-4} + bg(z)^{-2} + (2\nu+1)bg(z)^{-2} - b \\ &= -b + 2(\nu+1)bg(z)^{-2}. \end{aligned} \quad (3.8)$$

Equating the constant parts of equation (3.8), we get $b = -\wp(\varepsilon)$, and using the variable parts, we obtain the new potential as

$$\begin{aligned} V(z) &= 2(\nu+1)\wp(\varepsilon)g(z)^{-2}, \\ &= -\frac{i}{z}(\nu+1)\sqrt{\wp(\varepsilon)}. \end{aligned} \quad (3.9)$$

The function $f(z)$ can now be obtained from equation (2.11) as:

$$\begin{aligned} f(z) &= \left(b^{\frac{1}{2}} g(z)^{-1}\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\cdot 2z\sqrt{b}\right) \\ &= b^{-\frac{1}{4}} g(z)^{\frac{1}{2}} \exp\left(\frac{1}{2}\cdot 2iz\sqrt{\wp(\varepsilon)}\right) \\ &= i^{-\frac{1}{4}}\wp(\varepsilon)^{-\frac{1}{8}}\sqrt{\sqrt{2z}}\exp\left(iz\sqrt{\wp(\varepsilon)}\right). \end{aligned} \quad (3.10)$$

Now we use equations (3.9) and(3.10) to obtain new eigenfunction as

$$\begin{aligned} \psi(z) \equiv \psi_\nu(z) &= (g'(z))^{-\frac{1}{2}} \exp\left(\frac{1}{2}g(z)^2\right) H_\nu(g(z)) \\ &= \left(\frac{2z}{i\sqrt{\wp(\varepsilon)}}\right)^{\frac{1}{4}} \exp\left(iz\sqrt{\wp(\varepsilon)}\right) H_\nu\left(\sqrt{2iz\sqrt{\wp(\varepsilon)}}\right), \quad \nu \in \mathbb{Z}_0^+. \end{aligned}$$

Here, $\mathbb{Z}_0^+ := \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of counting numbers. We know (Riley *et al* [9], p. 627, §18.130) that the Hermite polynomial $H_\nu(g)$ satisfies the orthogonality condition

$$\begin{aligned} 1 = \langle \mathcal{N}_\nu H_\nu(g), \mathcal{N}_\nu H_\nu(g) \rangle &= \mathcal{N}_\nu^2 \int_{-\infty}^{\infty} |H_\nu(g)|^2 \mathrm{d}g = \mathcal{N}_\nu^2 2^\nu \nu! \sqrt{\pi}. \\ \therefore \qquad \mathcal{N}_\nu &= \sqrt{\frac{1}{2^\nu \nu! \sqrt{\pi}}}\,. \end{aligned}$$

The solution, in terms of the normalization constant $\mathcal{N}_\nu$ is given by

$$\begin{aligned} \Psi(z) &= \sum_{\nu=0}^{\infty} \mathcal{N}_\nu \psi_\nu(z) \\ &= \sum_{\nu=0}^{\infty} \sqrt{\frac{1}{2^\nu \nu! \sqrt{\pi}}} \left(\frac{2z}{i\sqrt{\wp(\varepsilon)}} \right)^{\frac{1}{4}} \exp\left(iz\sqrt{\wp(\varepsilon)} \right) H_\nu \left(\sqrt{2iz\sqrt{\wp(\varepsilon)}} \right). \end{aligned}$$

3.3. Exactly Solvable Potentials for Associated Laguerre Polynomials

In this subsection, we apply the formalism of Section 2 to the associated Laguerre polynomials $L_n^m(g)$ defined by

$$L_n^m(g) = \frac{e^g g^{-m}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}g^n} (g^{n+m} e^{-g}) = \sum_{k=0}^{n} (-1)^k \frac{(n+m)!}{k!(n-k)!(k+m)!} g^k,$$

where $n, m \in \mathbb{Z}_0^+$. These polynomials appear as solutions to the differential equation which we write in the form of equation (2.3) as

$$\ddot{F}(g(z)) - \left(\frac{m+1}{g(z)} - 1 \right) \dot{F}(g(z)) + \frac{n}{g(z)} F(g(z)) = 0 \qquad (3.11)$$

(cf: Riley *et al* [9], p.622,§18.120 and 18.121). The equation (3.11) has a regular singularity at $g = 0$ and an essential singularity at $g = \infty$. From equation (3.11), the coefficients $Q(g)$ and $R(g)$ are

$$Q(g) = \left(\frac{m+1}{g} - 1 \right) \qquad \text{and} \qquad R(g) = \frac{n}{g},$$

and $f(z)$ is

$$f(z) = (g'(z))^{-\frac{1}{2}} \exp\left(\frac{1}{2} \int^{g(z)} \left(\frac{m+1}{u} - 1 \right) \mathrm{d}u \right)$$

$$= \quad (g'(z))^{-\frac{1}{2}} g(z)^{\frac{m+1}{2}} \exp(\frac{1}{2} g(z)). \tag{3.12}$$

Substituting these expressions for $Q(g)$ and $R(g)$ into (2.14), we obtain

$$\begin{aligned}\wp(\varepsilon) - 2\wp(z) = {} & \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 - g''\left(\frac{m+1}{g} - 1\right) \\ & + (g')^2\left[\frac{n}{g} + \frac{1}{2}\frac{(m+1)g'}{g^2} - \frac{1}{4}\left(\frac{(m+1)^2}{g^2} - 2\frac{m+1}{g} + 1\right)\right].\end{aligned} \tag{3.13}$$

Let $\dfrac{g'(z)}{g(z)} = k$ (constant). Then, $g(z) = \exp(kz)$ and so

$$\wp(\varepsilon) - 2\wp(z) = -k^2 g^2 + k^2\left(m + n + \frac{5}{2}\right) g - k^2(m+1) + \frac{1}{4}(m+1)^2 k^2.$$

We now equate the constant parts of the last equation to get

$$4(m+1)k^2[(m+1) - 4] = 4\wp(\varepsilon)$$

so that

$$k = \sqrt{\frac{\wp(\varepsilon)}{(m+1)(m-3)}}.$$

The new exactly solvable potential, in this case, is given as

$$V(z) = \frac{\wp(\varepsilon)}{(m+1)(m-3)}\left[g(z)^2 - \left(m + n + \frac{5}{2}\right) g(z)\right].$$

The solution in this case is obtained as

$$\psi(z) \equiv \psi_n(z) = (g'(z))^{-\frac{1}{2}} g(z)^{\frac{m+1}{2}} \exp(\frac{1}{2} g(z)) L_n^m(g(z)).$$

The polynomials $L_n^m(g)$ satisfy the orthogonality condition

$$1 = \langle \mathcal{N}_n L_n^m(g)\ ,\ \mathcal{N}_n L_n^m(g)\rangle = \mathcal{N}_n^2 \int_0^\infty |L_n^m(g)|^2 g^m e^{-g} \mathrm{d}g = \mathcal{N}_n^2 \frac{(n+m)!}{n!}.$$

Thus,

$$\mathcal{N}_n = \sqrt{\frac{n!}{(n+m)!}}.$$

The normalized solution is, therefore, given as

$$\Psi(z) = \sum_{n=0}^{m} \sqrt{\frac{n!}{(n+m)!}} (g'(z))^{-\frac{1}{2}} g(z)^{\frac{m+1}{2}} \exp(\frac{1}{2} g(z)) L_n^m(g(z)),$$

where,

$$g(z) = \exp\left(z \sqrt{\frac{\wp(\varepsilon)}{(m+1)(m-3)}} \right).$$

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