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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

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Abstract: We prove the existence of multiple solutions for the fourth order differential equation $u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right)$ under certain nonhomogeneous boundary conditions. Here $\lambda > 0$, and h is assumed to be a nonnegative continuous function satisfying appropriate constraints. The proof involves transforming the fourth order problem into a system of two second order differential equations, and then using the Guo-Krasnosel'skii Fixed Point Theorem to establish the existence of at least three solutions.

AMS Subject Classification: 34B18
Key Words: differential equations, boundary value problem, multiple solutions

1. Introduction

In this paper, we prove the existence of at least three solutions to the following fourth order nonhomogeneous boundary value problem:

$$u^{(4)}(t) = \lambda h\left(t, u(t), u''(t)\right), \tag{1}$$

$$\alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -a, \tag{2}$$

$$\alpha_2 u''(0) - \gamma_2 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = b, \tag{3}$$

where $h : [0,1] \times [0,\infty) \times (-\infty, 0] \to [0,\infty)$ is nonnegative and continuous,

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and $a, b, \lambda, \alpha_i, \beta_i, \gamma_i, \delta_i > 0$ for $i = 1, 2$. The function h and the constants $a, b, \lambda, \alpha_i, \beta_i, \gamma_i,$ and δ_i are also required to satisfy additional conditions (see Corollary 4.2).

The proof involves transforming the system (1)-(3) into a system of two second order differential equations, and then making use of three applications of the Guo-Krasnosel'skii Fixed Point Theorem. This work is related to results and methods found in [1], [2], [3], [4], and [6].

2. Preliminaries

Let $u_1 = u$, $u_2 = -u_1''$, $g(t, u_1, u_2) = u_2$, and $h(t, u_1, -u_2) = f(t, u_1, u_2)$. Then (1)-(3) becomes

$$-u_2''(t) = \lambda f\left(t, u_1, u_2\right), \tag{4}$$

$$-u_1''(t) = g\left(t, u_1, u_2\right), \tag{5}$$

$$\alpha_1 u_1(0) - \gamma_1 u_1(1) = \beta_1 u_1'(0) - \delta_1 u_1'(1) = -a, \tag{6}$$

$$\alpha_2 u_2(0) - \gamma_2 u_2(1) = \beta_2 u_2'(0) - \delta_2 u_2'(1) = -b. \tag{7}$$

The system (4)-(7) can then be transformed into the equivalent homogeneous system

$$-u_2''(t) = \lambda f(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{8}$$

$$-u_1''(t) = g(t, u_1(t) + Q_1 t^2 + R_1, u_2(t) + Q_2 t^2 + R_2), \tag{9}$$

$$\alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0, \tag{10}$$

where $Q_i = \frac{a}{2\delta_i}$ and $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ for $i = 1, 2$. We now make the assumption that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$. This assumption, along with the previous assumptions on a, b, and λ, ensure that u and u'' are nonnegative and concave (post-substitution), that $R_i, Q_i > 0$, and that the Green's functions $G_k(t, s)$ (see below) are positive.

Solutions to (8)-(10) are of the form

$$u_2(t) = \lambda \int_0^1 G_2(t, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

$$u_1(t) = \int_0^1 G_1(t, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds,$$

where $G_k(t,s)$ are the Green's functions

$$G_k(t,s) = \frac{1}{M_k N_k} \begin{cases} \delta_k N_k t + \gamma_k M_k s + \gamma_k \beta_k, & 0 \le t \le s \le 1, \\ \beta_k N_k t + \alpha_k M_k s + \gamma_k \beta_k, & 0 \le s \le t \le 1, \end{cases}$$

and $M_k = \delta_k - \beta_k$, $N_k = \alpha_k - \gamma_k$ for $k = 1,2$. Note that $G_k(t,s)$ is positive, and hence the solutions are positive provided they exist. Also, we have

$$\max_{t\in[0,1]} \int_0^1 G_k(t,s)\,ds = \frac{\alpha_k\,(\delta_k + \beta_k)}{2M_k N_k}$$

and

$$\max_{t\in[0,1]} \int_0^1 \frac{\partial}{\partial t} G_k(t,s)\,ds = \frac{\delta_k}{M_k}.$$

We assume the following hypotheses are satisfied for the functions f and g:

(H0) $f, g : [0,1] \times [0,\infty)^2 \to [0,\infty)$ are continuous functions that are nondecreasing in their last two variables.

(H1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0,\infty)^2$ with $x_1 + x_2 \neq 0$, there exists $\kappa > 0$ such that $f(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.

(H2) Let $z = x_1 + x_2$. Then

$$\lim_{z\to 0^+} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

(H3) Let $z = x_1 + x_2$. Then

$$\lim_{z\to \infty} \frac{f(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0,1]$.

(H4) There exists a $0 < \zeta < \frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}$ and $q > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0,\infty)^2$ with $0 < \overline{x}_1 + \overline{x}_2 < q$, we have $g(t, \overline{x}_1, \overline{x}_2) \le \zeta(\overline{x}_1 + \overline{x}_2)$ for each $t \in [0,1]$.

(H5) There exists a $0 < \theta < \frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}$ and $r > 0$ such that, for all $(\overline{x}_1, \overline{x}_2) \in [0,\infty)^2$ with $\overline{x}_1+\overline{x}_2 > r$, we have $g(t, \overline{x}_1, \overline{x}_2) \leq \theta(\overline{x}_1+\overline{x}_2)$ for each $t \in [0, 1]$.

Let $(X, \|\cdot\|)$ be the Banach space $X = C^1([0, 1]; \mathbb{R}) \times C^1([0, 1]; \mathbb{R})$ endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_\infty + \|u_2\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$\begin{aligned} C = \{(u_1, u_2) \in X \mid\ & u_i \text{ is nonnegative and concave;} \\ & \alpha_i u_i(0) - \gamma_i u_i(1) = \beta_i u_i'(0) - \delta_i u_i'(1) = 0 \text{ for } i = 1, 2\}. \end{aligned}$$

Next, let Ω_ρ denote the open set $\Omega_\rho = \{(u_1, u_2) \in X : \|(u_1, u_2)\| < \rho\}$. Lastly, define $T : X \to X$ to be the operator

$$T(u_1, u_2) = (A_1(u_1, u_2), A_2(u_1, u_2)),$$

where

$$A_2(u_1, u_2)(t) = \lambda \int_0^1 G_2(t, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds$$

and

$$A_1(u_1, u_2)(t) = \int_0^1 G_1(t, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds.$$

The following two lemmas will be useful, but we omit the proofs here. Lemma 2.1 can be obtained by applying the Arzela-Ascoli Theorem, and the proof of Lemma 2.2 is a straightforward exercise.

Lemma 2.1. *$T : C \to C$ is a completely continuous operator.*

Lemma 2.2. *Let $u(t)$ be a nonnegative concave function which is continuous on $[0, 1]$. Then for all $\alpha, \beta \in (0, 1)$, with $\alpha < \beta$, we have*

$$\inf_{t\in[\alpha,\beta]} u(t) \geq \alpha(1 - \beta)\|u\|_\infty.$$

3. Technical Results

Lemma 3.1. *Suppose* (H0) *and* (H1) *hold, and let* $\rho^* > 0$. *Then there exists* Λ *such that, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$, *we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.

Proof. Let $\rho^* > 0$ and let $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$. Assume α and β are as in (H1) and set $r = \alpha(1 - \beta)$. Define

$$K = \inf\left\{\frac{f(t, rc_1, rc_2)}{r(c_1 + c_2)} : c_1, c_2 \geq 0, c_1 + c_2 = p^*, t \in [\alpha, \beta]\right\}.$$

The existence of a positive K follows from assumption (H1).

Now set $\Lambda \geq \left[Kr \int_\alpha^\beta G_2(1, s)ds\right]^{-1}$. Utilizing Lemma 2.2, for $t \in [\alpha, \beta]$ we know that $u_i(t) + Q_i t^2 + R_i \geq r\|u_i\|_\infty$ for $i = 1, 2$. Pairing this with the nondecreasing properties of f, it follows that

$$\begin{aligned}
\|T(u_1, u_2)\| &\geq \|A_2(u_1, u_2)\|_\infty \\
&\geq \lambda \int_0^1 G_2(1, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2)ds \\
&\geq \lambda \int_\alpha^\beta G_2(1, s) f(s, r\|u_1\|_\infty, r\|u_2\|_\infty)ds \\
&= \lambda r\|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s)\frac{f(s, r\|u_1\|_\infty, r\|u_2\|_\infty)}{r\|(u_1, u_2)\|}ds \\
&\geq \lambda K r\|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s)ds \\
&\geq \Lambda K r\|(u_1, u_2)\| \int_\alpha^\beta G_2(1, s)ds \\
&\geq \|(u_1, u_2)\|
\end{aligned}$$

for $\lambda \geq \Lambda$, which completes the proof. □

Lemma 3.2. *Fix* $\Lambda > 0$, *and suppose* (H0) *and* (H1) *hold. Then, for every* $\lambda \geq \Lambda$ *and* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$, *there exists* $\rho_1 = \rho_1(\Lambda, Q_1, Q_2, R_1, R_2)$ *such that, for every* $\rho \leq \rho_1$, *we have*

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_\rho$.

Proof. By (H1) and the nondecreasing property of f, there exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, and $\kappa > 0$ such that for all $t \in [\alpha, \beta]$ and for all $c_1, c_2 \geq 0$,

$$f(t, c_1 + Q_1 t^2 + R_1, c_2 + Q_2 t^2 + R_2) \geq f(t, Q_1\alpha^2, Q_2\alpha^2) > \kappa.$$

Take $\rho_1 = \Lambda\kappa \int_\alpha^\beta G_2(1,s)ds$. Then, for all $(u_1, u_2) \in C \cap \partial\Omega_\rho$ where $\rho \leq \rho_1$, we have

$$\begin{aligned}
\|T(u_1,u_2)\| &\geq \|A_2(u_1,u_2)\|_\infty \\
&\geq \lambda \int_0^1 G_2(1,s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\
&\geq \lambda \int_\alpha^\beta G_2(1,s) f(s, Q_1\alpha^2, Q_2\alpha^2) ds \\
&> \lambda\kappa \int_\alpha^\beta G_2(1,s) ds \\
&= \lambda\kappa \|(u_1,u_2)\| \int_\alpha^\beta \frac{G_2(1,s)}{\|(u_1,u_2)\|} ds \\
&\geq \frac{\rho_1}{\rho} \|(u_1,u_2)\| \\
&\geq \|(u_1,u_2)\|. \qquad \square
\end{aligned}$$

Lemma 3.3. *Suppose* (H0), (H2), *and* (H4) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there exists* $\rho_2 \in (0, \rho^*)$ *and* $\overline{\zeta} > 0$ *such that for every* $(Q_1, Q_2, R_1, R_2) \in [0, \infty)^4$ *with* $0 < Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$, *we have*

$$\|T(u_1,u_2)\| \leq \|(u_1,u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon < \frac{M_2 N_2}{\alpha_2(\delta_2 + \beta_2)}$. From (H2), there exists $\overline{\rho}_2 \in (0, \rho^*)$ such that for $x_1 + x_2 = \overline{\rho}_2$ with $(x_1, x_2) \in [0, \infty)^2$ and for $Q_1 + Q_2 + R_1 + R_2 \leq \overline{\rho}_2$, we have

$$f(t, x_1 + Q_1 + R_1, x_2 + Q_2 + R_2) \leq \epsilon \left[(x_1 + Q_1 + R_1) + (x_2 + Q_2 + R_2)\right]$$

for $t \in [0,1]$.

By (H4) there exists $\zeta > 0$ satisfying $\zeta < \frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}$ and there exists $q > 0$ such that for $(x_1+Q_1+R_1, x_2+Q_2+R_2) \in [0,\infty)^2$ with $x_1+Q_1+R_1+x_2+Q_2+R_2 < q$ we have

$$g(t, x_1 + Q_1 + R_1, x_2 + Q_2 + R_2) \leq \zeta\left[(x_1 + Q_1 + R_1) + (x_2 + Q_2 + R_2)\right]$$

for $t \in [0, 1]$.
Let $0 < \rho_2 < \min(q/2, \overline{\rho}_2)$. Take $(u_1, u_2) \in C \cap \partial\Omega_{\rho_2}$, and $Q_1 + Q_2 + R_1 + R_2 \leq \rho_2$. Then by (H0) and above we have

$$\begin{aligned} A_2(u_1, u_2) &= \lambda \int_0^1 G_2(t, s) f(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\ &\leq \lambda \int_0^1 G_2(t, s) f(s, \|u_1\|_\infty + Q_1 + R_1, \|u_2\|_\infty + Q_2 + R_2) ds \\ &\leq \lambda\epsilon\left[\|(u_1, u_2)\| + Q_1 + Q_2 + R_1 + R_2\right] \int_0^1 G_2(t, s) ds \\ &\leq 2\lambda\epsilon\|(u_1, u_2)\| \int_0^1 G_2(t, s) ds \\ &\leq \lambda\epsilon \frac{\alpha_2\left(\delta_2 + \beta_2\right)}{M_2 N_2} \|(u_1, u_2)\| \end{aligned}$$

for $t \in [0, 1]$.
Note that $\|u_1\|_\infty + \|u_2\|_\infty + Q_1 + Q_2 + R_1 + R_2 \leq 2\rho_2 < q$. So

$$g(t, \|u_1\|_\infty + Q_1 + R_1, \|u_2\|_\infty + Q_2 + R_2) \leq \zeta(\|u_1\|_\infty + \|u_2\|_\infty + Q_1 + Q_2 + R_1 + R_2).$$

Pick $\zeta' < 1$, and suppose $Q_1 + Q_2 + R_1 + R_2 < \zeta'\rho_2$. Let $\overline{\zeta} = \zeta'\rho_2$. We have, by (H0) and above,

$$\begin{aligned} A_1(u_1, u_2) &= \int_0^1 G_1(t, s) g(s, u_1(s) + Q_1 s^2 + R_1, u_2(s) + Q_2 s^2 + R_2) ds \\ &\leq \zeta\left[\|(u_1, u_2)\| + Q_1 + Q_2 + R_1 + R_2\right] \int_0^1 G_1(t, s) ds \\ &\leq \zeta\left(1 + \zeta'\right) \|(u_1, u_2)\| \int_0^1 G_1(t, s) ds \\ &\leq \zeta\left(1 + \zeta'\right) \frac{\alpha_1\left(\delta_1 + \beta_1\right)}{2M_1 N_1} \|(u_1, u_2)\| \end{aligned}$$

for $t \in [0, 1]$.

Thus,

$$\|T(u_1,u_2)\| \le \left[\zeta\left(1+\zeta'\right)\frac{\alpha_1\left(\delta_1+\beta_1\right)}{2M_1N_1} + \lambda\epsilon\frac{\alpha_2\left(\delta_2+\beta_2\right)}{M_2N_2}\right]\|(u_1,u_2)\|$$

for $(u_1,u_2)\in C\cap\Omega_{\rho_2}$ and $(Q_1,Q_2,R_1,R_2)\in[0,\infty)^4$ with $0<Q_1+Q_2+R_1+R_2<\overline{\zeta}$. Picking ϵ and ζ' small enough gives the desired result. □

Lemma 3.4. *Suppose* $0<Q_1+Q_2+R_1+R_2<\overline{\zeta}$, *where* $\overline{\zeta}>0$ *is given. Suppose further that assumptions* (H0), (H3), *and* (H5) *hold. Then, for every* $\lambda>0$, *there exists* $\rho_3=\rho_3(\overline{\zeta},\lambda)$ *such that for every* $\rho\ge\rho_3$, *we have*

$$\|T(u_1,u_2)\|\le\|(u_1,u_2)\|$$

for each $(u_1,u_2)\in C\cap\partial\Omega_\rho$.

Proof. Let $\overline{\zeta}>0$ be given and let $Q_1+Q_2+R_1+R_2<\overline{\zeta}$. Note that by (H5) there exists $0<\theta<\frac{2M_1N_1}{\alpha_1(\delta_1+\beta_1)}$ and $r>0$ such that for all $(x_1+Q_1+R_1,x_2+Q_2+R_2)\in[0,\infty)^2$ with $x_1+Q_1+R_1+x_2+Q_2+R_2>r$, we have $g(t,x_1+Q_1+R_1,x_2+Q_2+R_2)\le\theta(x_1+Q_1+R_1+x_2+Q_2+R_2)$ for $t\in[0,1]$. Let $\epsilon>0$ and pick q_1 large enough so that $q_1+(Q_1+Q_2+R_1+R_2)>r$ and $\epsilon>\frac{\theta\overline{\zeta}}{q_1}$. Then, for $(u_1,u_2)\in C\cap\partial\Omega_{q_1}$,

$$\begin{aligned}
g(t,\|u_1\|_\infty+Q_1+R_1,\|u_2\|_\infty+Q_2+R_2) \le&\theta\left[(\|u_1\|_\infty+Q_1+R_1)\right.\\
&\left.+(\|u_2\|_\infty+Q_2+R_2)\right]\\
\le&\theta(\|u_1\|_\infty+\|u_2\|_\infty)\\
&+\theta(Q_1+Q_2+R_1+R_2)\\
\le&\theta(\|u_1\|_\infty+\|u_2\|_\infty)\\
&+\epsilon(\|u_1\|_\infty+\|u_2\|_\infty)\\
\le&(\epsilon+\theta)\|(u_1,u_2)\|.
\end{aligned}$$

Applying the above with the nondecreasing property of g we see that for $t\in[0,1]$,

$$\begin{aligned}
A_1(u_1,u_2) &= \int_0^1 G_1(t,s)g(s,u_1(s)+Q_1s^2+R_1,u_2(s)+Q_2s^2+R_2)ds\\
&\le \int_0^1 G_1(t,s)g(s,\|u_1\|_\infty+Q_1+R_1,\|u_2\|_\infty+Q_2+R_2)ds\\
&\le (\epsilon+\theta)\|(u_1,u_2)\|\int_0^1 G_1(t,s)ds
\end{aligned}$$

$$\leq (\epsilon + \theta) \frac{\alpha_1 (\delta_1 + \beta_1)}{2M_1 N_1} \|(u_1, u_2)\|.$$

Now, let $\eta > 0$. Then by (H0) and (H3), there exists $q_2 > 0$ such that for every $(x_1, x_2) \in [0, \infty)^2$ with $x_1 + x_2 \geq q_2$, we have

$$f(t, x_1 + Q_1 + R_1, x_2 + Q_2 + R_2) \leq \eta \left[(x_1 + Q_1 + R_1) + (x_2 + Q_2 + R_2) \right].$$

Let $q_3 = \max \left\{ \overline{\zeta}, q_2 \right\}$ and $(u_1, u_2) \in C \cap \partial\Omega_{q_3}$. Recalling $Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$, it follows

$$\begin{aligned} f(t, \|u_1\|_\infty + Q_1 + R_1, \|u_2\|_\infty + Q_2 + R_2) &\leq \eta(\|u_1\|_\infty + \|u_2\|_\infty) + \overline{\zeta}\eta \\ &\leq 2\eta \|(u_1, u_2)\|. \end{aligned}$$

Combining this with the nondecreasing properties of f, we have

$$\begin{aligned} A_2(u_1, u_2) &\leq 2\eta\lambda \|(u_1, u_2)\| \int_0^1 G_2(t, s) ds \\ &\leq \eta\lambda \frac{\alpha_2 (\delta_2 + \beta_2)}{M_2 N_2} \|(u_1, u_2)\|. \end{aligned}$$

Take $\rho_3 = \max \{q_1, q_3\}$, and let $\rho \geq \rho_3$. Then, given $(u_1, u_2) \in C \cap \partial\Omega_\rho$, we have

$$\|T(u_1, u_2)\| \leq \left[(\epsilon + \theta) \frac{\alpha_1 (\delta_1 + \beta_1)}{2M_1 N_1} + \eta\lambda \frac{\alpha_2 (\delta_2 + \beta_2)}{M_2 N_2} \right] \|(u_1, u_2)\|.$$

Picking ϵ and η small enough gives

$$\|T(u_1, u_2)\| \leq \|(u_1, u_2)\|$$

as needed. □

4. Main Results

Theorem 4.1. *Suppose hypotheses* (H0)-(H5) *are satisfied for functions f and g. Suppose additionally that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$. Then there exists $\Lambda > 0$ such that given $\lambda \geq \Lambda$, there exists $\overline{\zeta} > 0$ such that for every $a, b > 0$ satisfying $0 < Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$, the system* (8)-(10) *has at least three positive solutions.*

Proof. Suppose f and g satisfy hypotheses (H0)-(H5) and that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ for $i = 1, 2$. Let $\rho^* > 0$ be fixed. By Lemma 3.1, there exists $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $a, b > 0$,

$$\|T(u_1, u_2)\| \geq \|(u_1, u_2)\|$$

for each $(u_1, u_2) \in C \cap \partial\Omega_{\rho^*}$.
Fix $\lambda \geq \Lambda$. By Lemmas 3.2 through 3.4, there exists $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that for $a, b > 0$ satisfying $Q_1+Q_2+R_1+R_2 < \overline{\zeta}$,

$$\begin{aligned}
\|T(u_1, u_2)\| &\geq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_1}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_2}, \\
\|T(u_1, u_2)\| &\leq \|(u_1, u_2)\|, \text{ for } (u_1, u_2) \in C \cap \partial\Omega_{\rho_3}.
\end{aligned}$$

After applying the Guo-Krasnosel'skii Fixed Point Theorem three times, we get that there exist three positive solutions (u_1, u_2), (v_1, v_2),$(w_1, w_2) \in C$ such that

$$\rho_1 < \|(u_1, u_2)\| < \rho_2 < \|(v_1, v_2)\| < \rho^* < \|(w_1, w_2)\| < \rho_3. \qquad \square$$

After making the appropriate transformations (see the Preliminaries section), the following corollary is obtained. We note that the additional constraint on the constants $a, b, \alpha_i, \beta_i, \gamma_i$, and δ_i (see (11) below) is needed to ensure that hypotheses corresponding to (H4) and (H5) are satisfied. This new constraint allows for the removal of any hypotheses corresponding to (H4) and (H5).

Corollary 4.2. *Suppose the following hypotheses are satisfied for a function $h : [0, 1] \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$:*

(H0′) *h is continuous, nondecreasing in its second variable, and nonincreasing in its third variable.*

(H1′) *There exists $\alpha, \beta \in (0, 1)$, $\alpha < \beta$, such that, given $(x_1, x_2) \in [0, \infty) \times (-\infty, 0] \setminus \{(0, 0)\}$, there exists $\kappa > 0$ such that $h(t, x_1, x_2) > \kappa$ for $t \in [\alpha, \beta]$.*

(H2′) *Let $z = x_1 - x_2 > 0$. Then*

$$\lim_{z\to 0^+} \frac{h(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0, 1]$.

(H3′) Let $z = x_1 - x_2 > 0$. *Then*

$$\lim_{z \to \infty} \frac{h(t, x_1, x_2)}{z} = 0$$

uniformly for $t \in [0, 1]$.

Suppose also that $\alpha_i > \gamma_i \geq 2\delta_i > \delta_i > \beta_i > 0$ *for* $i = 1, 2$. *Then there exists* $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$, *there exists* $\overline{\zeta} > 0$ *such that for every* $a, b > 0$ *that satisfies the properties that after setting* $Q_i = \frac{a}{2\delta_i}$ *and* $R_i = -\frac{a(2\delta_i - \gamma_i)}{2\delta_i(\alpha_i - \gamma_i)}$ *for* $i = 1, 2$, *we have* $0 < Q_1 + Q_2 + R_1 + R_2 < \overline{\zeta}$ *and*

$$Q_2 + R_2 < \frac{2(\delta_1 - \beta_1)(\alpha_1 - \gamma_1)}{\alpha_1(\delta_1 + \beta_1)} \left(Q_1 + R_1 + Q_2 + R_2\right), \tag{11}$$

the system (1)-(3) *has at least three positive solutions.*

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