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STABILITY OF THE DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND NON FIXED IMPULSIVE MOMENTS USING SEQUENCES OF LYAPUNOV'S FUNCTIONS

R.B. Chukleva[1 §], A.B. Dishliev[2], K.G. Dishlieva[3]

[1]Technical University of Plovdiv
Plovdiv, BULGARIA

Department of Mathematics
University of Chemical Technology and Metallurgy
Sofia, BULGARIA

[3]Technical University of Sofia
Sofia, BULGARIA

Abstract: A specific class of non-linear non-autonomous systems ordinary differential equations with variable structure and impulses are studied in the paper. The change of the system right side and impulsive effects of the solution are realized at the moments, at which the so-called switching functions, defined in the system phase space, are canceled. Sufficient conditions for stability, uniform stability and uniform asymptotically stability of zero solution for the systems investigated are obtained. The results are received using a modification of the Lyapunov's Second Method. For this purpose, there are introduced sequences of the scalar piecewise - continuous functions of the Lyapunov class. The consecutive change (activation) of the Lyapunov's functions from the sequence are synchronized with the changing of the structure of the system investigated. Note that, it is allowed each one of the Lyapunov's functions of the sequence to be piecewise - continuous. The points of discontinuity coincide with the points of the set of switching of the corresponding right side of the system.

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[§]Correspondence author

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1. Introduction

S. Gurgula and N. Perestyuk are the first, which apply the Lyapunov's Second Method to study the solutions of the impulsive systems. They used the classical continuous Lyapunov's functions to investigate the stability of zero solution for such systems in [35]. We note that using continuous auxiliary functions reduces strongly the application of the direct method. The main reason for this is the fact that the solutions of the impulsive systems are continuous piecewise functions. The piecewise Lyapunov's functions are introduced by Bainov and Simeonov in [16]. On the theory of stability of impulsive systems are devoted number of studies, we will cite the following: [4], [5], [10], [11], [12], [13], [15], [16], [17], [18], [22], [23], [24], [25], [26], [27], [28], [29], [34], [35], [37], [39], [40], [41], [44], [47], [48], [50], [51] and [52]. The applications of impulsive equations are numerous. We point out the articles: [1], [2], [3], [14], [26], [28], [29], [30], [36], [42], [43], [49], [52] and [53]. The equations with variable structure are applicable mainly in the control theory [6], [7], [8], [9], [19], [20], [21], [32], [33], [38], [45] and [46].

The main object of investigation is the following initial problem for nonlinear non-autonomous ordinary differential equations with variable structure and impulses in non-fixed moments:

$$\frac{dx}{dt} = f_i\left(t,x,\right), \qquad \varphi_i\left(x\left(t\right)\right) \neq 0,\ \ t_{i-1} < t < t_i, \tag{1}$$

$$\varphi_i\left(x\left(t_i\right)\right) = 0, \qquad i = 1,2,\dots, \tag{2}$$

$$x\left(t_i+0\right) = x\left(t_i\right) + I_i\left(x\left(t_i\right)\right), \tag{3}$$

$$x\left(t_0\right) = x_0, \tag{4}$$

where:

- The functions $f_i \colon \mathbb{R}^+ \times D \to \mathbb{R}^n$;
- The phase space of the system considered D is non empty domain of $\mathbb{R}^n$;
- The functions $\varphi_i \colon D \to \mathbb{R}$;
- The functions $I_i \colon D \to \mathbb{R}^n$;
- $(\mathrm{I}_d + I_i) : D \to D$, Id is the identity in $\mathbb{R}^n$;

- The initial point $(t_0, x_0) \in \mathbb{R}^+ \times D$.

Each one of the functions φ_i and I_i is called switching and impulsive, respectively, $i = 1, 2, \ldots$. We denote by $x\,(t; t_0, x_0)$ the solution of problem (1), (2), (3), (4). Then

$$x(t;t_0,x_0) = x(t;t_0,x_0,f_1,\varphi_1,I_1,f_2,\varphi_2,I_2,\ldots)$$
$$= \begin{cases} x\,(t;t_0,x_0,f_1)\,, & t_0 < t < t_1; \\ x\,(t;t_0,x_0,f_1,\varphi_1,I_1,f_2)\,, & t_1 < t < t_2; \\ \vdots & \\ x\,(t;t_0,x_0,f_1,\varphi_1,I_1,f_2,\ldots,f_i,\varphi_i,I_i,f_{i+1})\,, & t_i < t < t_{i+1}; \\ \vdots & \end{cases} \tag{5}$$

We introduce the notations:

- $\Phi_i = \{x \in D; \varphi_i\,(x) = 0\}$, $i = 1, 2, \ldots$, are switching hypersurfaces of the problem considered;
- $\Phi = \bigcup\limits_{i=1,2,\ldots} \Phi_i$;
- $\gamma\,(t_0, x_0) = \{x\,(t; t_0, x_0)),\ t_0 \le t < \infty\}$ is the trajectory of the problem for $t_0 \le t < \infty$;
- $B_\delta(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$.

2. Preliminary Remarks

We use the following conditions:

H1. $0 \in D$ and $f_i\,(t, 0) = 0$, $t \in \mathbb{R}^+$, $i = 1, 2, \ldots$.

H2. $\varphi_i\,(0) = \varphi_i\,(0, 0, \ldots, 0) \neq 0$, $i = 1, 2, \ldots$, i.e. $0 = (0, 0, \ldots, 0) \notin \Phi$.

H3. The functions $f_i \in C[\mathbb{R}^+ \times D, \mathbb{R}^n]$, $i = 1, 2, \ldots$.

H4. The constant $C_f > 0$ exists, such that

$$\big(\forall\,(t,x) \subset \mathbb{R}^+ \times D\big) \Rightarrow \parallel f_i\,(t,x) \parallel \le C_f, \qquad i = 1, 2, \ldots.$$

H5. The functions $\varphi_i \in C^1[D, \mathbb{R}]$, $i = 1, 2, \ldots$.

H6. The constant $C_{\mathrm{grad}\varphi} > 0$ exists, such that

$$(\forall x \in D) \Rightarrow \| \operatorname{grad}\varphi_i(x) \| \leq C_{\mathrm{grad}\varphi}, \qquad i = 1, 2, \ldots.$$

H7. The functions $I_i \in C[D, \mathbb{R}^n]$ and $(\mathrm{Id} + I_i) : \Phi_i \to D$, $i = 1, 2, \ldots$.

H8. The constant $C_{\varphi(\mathrm{Id}+I)} > 0$ exists, such that

$$(\forall x \in \Phi_i) \Rightarrow | \varphi_{i+1}((\mathrm{Id} + I_i)(x)) | = | \varphi_{i+1}(x + I_i(x)) | \geq C_{\varphi(\mathrm{Id}+I)}, \qquad i = 1, 2, \ldots.$$

H9. The next inequalities are valid:

$$\varphi_{i+1}((\mathrm{Id} + I_i)(x)) \langle \operatorname{grad}\varphi_{i+1}(x), f_{i+1}(t, x) \rangle < 0, \qquad (t, x) \in \mathbb{R}^+ \times D, \; i = 1, 2, \ldots.$$

H10. For every point $(t_0, x_0) \in \mathbb{R}^+ \times D$ and for each $i = 1, 2, \ldots$ the solution of the initial problem

$$\frac{dx}{dt} = f_i(t, x), \qquad x(t_0) = x_0$$

exists and it is unique for $t \geq t_0$.

The following theorems are valid.

Theorem 1. *The conditions H1, H2 and H3 are fulfilled.*
Then the impulsive system (1), (2), (3) possesses a zero solution, i.e. $x(t; t_0, 0) = 0$ *for* $0 \leq t_0 \leq t < \infty$.

The following two theorems will be useful for the subsequent consideration.

Theorem 2. (see [20]) *The conditions H3–H9 are fulfilled.*
Then:

1. *If the trajectory* $\gamma(t_0, x_0)$ *of problem (1), (2), (3), (4) meets consecutively the switching hypersurfaces* Φ_i *and* Φ_{i+1}*, then the following estimate is valid for the corresponding switching moments* t_i *and* t_{i+1}*:*

$$t_{i+1} - t_i \geq \frac{C_{\varphi(\mathrm{Id}+I)}}{C_{\mathrm{grad}\varphi} . C_f}.$$

2. *If the trajectory $\gamma(t_0, x_0)$ meets all the switching hypersurfaces Φ_i, $i = 1, 2, \ldots$, then the moments of switching grow up boundlessly, i.e.* $\lim\limits_{i\to\infty} t_i = \infty$.

Theorem 3. (see [20]) *The conditions H3–H10 are fulfilled:*
Then the solution $x(t; t_0, x_0)$ of problem (1), (2), (3), (4) exists and it is unique for $t_0 \le t < \infty$.

The following three definitions are fundamental for the research in this article.

Definition 1. We say that the zero solution of problem (1), (2), (3) is:

- *stable* on the initial condition, if:

$$\left(\forall t_0 \in \mathbb{R}^+\right)(\forall \varepsilon = \text{const} > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0):$$
$$(\forall x_0 \in D, \| x_0 \| < \delta) \;\Rightarrow \| x(t; t_0, x_0) \| < \varepsilon, \; t \ge t_0;$$

- *uniformly stable* on the initial condition, if:

$$\left(\forall t_0 \in \mathbb{R}^+\right)(\forall \varepsilon = \text{const} > 0)(\exists \delta = \delta(\varepsilon) > 0):$$
$$\left(\forall x_0 \in D \bigcap B_\delta(0)\right) \;\Rightarrow \| x(t; t_0, x_0) \| < \varepsilon, t \ge t_0;$$

- *asymptotically stable* on the initial condition, if it is stable and

$$\left(\forall t_0 \in \mathbb{R}^+\right)(\exists \lambda = \lambda(t_0) > 0): \left(\forall x_0 \in D \bigcap B_\lambda(0)\right) \Rightarrow \lim_{i\to\infty} x(t; t_0, x_0) = 0;$$

- *uniform asymptotically stable* on the initial condition, if it is uniformly stable and

$$(\exists \lambda = \text{const} > 0)\left(\forall t_0 \in \mathbb{R}^+\right)(\forall \varepsilon = \text{const} > 0)(\exists T = T(\varepsilon) > 0):$$
$$(\forall t \ge t_0 + T)(\forall x_0 \in B_\lambda(0)) \;\Rightarrow \| x(t; t_0, x_0) \| < \varepsilon.$$

Definition 2. We say that the *sequence of scalar piecewise continuous Lyapunov's functions* is given:

$$\left\{V_i; V_i \colon \mathbb{R}^+ \times D \to \mathbb{R}^+, i = 1, 2, \ldots\right\},$$

corresponding to the system of differential equations with variable structure and impulses (1), (2), (3), if:

1. $V_i \in C[\mathbb{R}^+ \times D \setminus \Phi_i, \mathbb{R}^+]$, $i = 1, 2, \ldots$;
2. For every point $(t, x_{\Phi_i}) \in \mathbb{R}^+ \times \Phi_i$ and each $i = 1, 2, \ldots$ there exist limits:

$$\lim_{\substack{x \to x_{\Phi_i},\ \Phi_i(x)<0, \\ (t,x)\in\mathbb{R}^+\times D}} V_i(t, x) = V_i(t, x_{\Phi_i} - 0) = V_i(t, x_{\Phi_i}),$$
$$\lim_{\substack{x \to x_{\Phi_i},\ \Phi_i(x)>0, \\ (t,x)\in\mathbb{R}^+\times D}} V_i(t, x) = V_i(t, x_{\Phi_i} + 0).$$

We note that the equalities in the definition above

$$V_i(t, x_{\Phi_i} - 0) = V_i(t, x_{\Phi_i}), \quad (t, x_{\Phi_i}) \in \mathbb{R}^+ \times \Phi_i,\ i = 1, 2, \ldots$$

are given for "completeness" of the statement. Without loss of generality we assume that

$$V_i(t, x_{\Phi_i} + 0) = V_i(t, x_{\Phi_i}), (t, x_{\Phi_i}) \in \mathbb{R}^+ \times \Phi_i, \quad i = 1, 2, \ldots.$$

Definition 3. Let $\{V_i; i = 1, 2, \ldots\}$ is a sequence of scalar piecewise continuous Lyapunov's functions.

Then for every point $(t, x) \in \mathbb{R}^+ \times D \setminus \Phi_i$ and for each $i = 1, 2, \ldots$ we define a *derivative of Lyapunov's function* V_i at the point (t, x) of impulsive system (1), (2), (3):

$$\dot{V}_i(t, x) = \dot{V}_{i,(1),(2),(3)}(t, x) = \lim_{h\to 0} \frac{1}{h} \left(V_i(t + h, x + hf_i(t, x)) - V_i(t, x) \right).$$

Remark 1. It can be shown that for every point

$$(t, x) = (t, x(t; t_0, x_0) \in [t_0, \infty) \times D \setminus \Phi_i$$

and for each $i = 1, 2, \ldots$ it is satisfied

$$\begin{aligned} \dot{V}_i(t, x) &= D^+_{(1),(2),(3)} V_i(t, x) \\ &= D^+_{(1),(2),(3)} V_i(t, x(t; t_0, x_0)) \\ &= \limsup_{h\to 0} \frac{1}{h} \left(V_i(t + h, x(t + h; t_0, x_0)) - V_i(t, x(t; t_0, x_0)) \right). \end{aligned}$$

In other words, the derivative of each of the Lyapunov's functions V_i, $i = 1, 2, \ldots$, in every point

$$(t, x) = (t, x(t; t_0, x_0)) \in (t_0, \infty) \times D \setminus \Phi_i$$

of the impulsive system of differential equations (1), (2), (3) coincides with the upper right Dini derivative at the same point on the solution of the system considered.

Further we will use the following class of scalar functions:

$$K = \left\{a \in C[\mathbb{R}^+, \mathbb{R}^+],\ a \text{ is strictly monotonically increasing and } a\,(0) = 0\right\}.$$

3. Main Results

The main purpose of this paragraph is finding the sufficient conditions for some types of stability of the zero solution of system (1) (2) (3) by means of sequences of scalar piecewise continuous Lyapunov's functions.

Theorem 4. *Let the following conditions hold:*

1. *The conditions H1-H9 are valid.*
2. *There exists a sequence of scalar piecewise continuous Lyapunov's functions*
$$\left\{V_i; V_i \colon \mathbb{R}^+ \times D \to \mathbb{R}^+,\ i = 1, 2, \ldots\right\},$$
corresponding to the impulsive system of differential equations (1), (2), (3), such that:

   (2.1) $V_i\,(t, 0) = 0,\ t \in \mathbb{R}^+,\ i = 1, 2, \ldots;$

   (2.2) To the upper sequence of Lyapunov's functions
$$\{V_i; i = 1, 2, \ldots\},$$
a function $a \in K$ corresponds, such that
$$a\,(\|\ x\ \|) \leq V_i\,(t, x)\,, (t, x) \in \mathbb{R}^+ \times D, i = 1, 2, \ldots;$$

   (2.3) It is satisfied
$$V_{i+1}\,(t + 0, x + I_i\,(x)) = V_{i+1}\,(t, x + I_i\,(x)) \leq V_i\,(t, x)\,, \qquad (t, x) \in \mathbb{R}^+ \times \Phi_i,\ i = 1, 2, \ldots;$$

(2.4) It is satisfied

$$\dot{V}_i\,(t,x) \le 0, \quad (t,x) \in \mathbb{R}^+ \times (D \diagdown \Phi_i)\,,\ i = 1,2,\ldots$$

(2.5) The sequence of Lyapunov's functions $V_1, V_2, \ldots$ is equipotential (of the same degree) continuous at the point $0 \in D \diagdown \Phi$, uniformly in the argument $t \in \mathbb{R}^+$, i.e.

$$\begin{aligned}&(\forall \varepsilon > 0)\,(\exists \delta = \delta\,(\varepsilon) > 0) : \\ &(\forall\,(t,x) \in \mathbb{R}^+ \times (D \cap B_\delta\,(0)))\,(\forall i = 1,2,\ldots) \\ &\Rightarrow \mid V_i\,(t,x) - V_i\,(t,0) \mid = V_i\,(t,x) < \varepsilon.\end{aligned}$$

Then the initial problem of impulsive system differential equations (1), (2), (3), (4) possesses a solution continuable for all $t \ge t_0$.

Proof. According to Theorem 1, the zero solution of the system differential equations with variable structure and impulses (1), (2), (3) exists for $0 \le t_0 \le t < \infty$.

Since $0 \in D$ and D is a domain, i.e. open set, then, it follows that

$$(\exists \alpha = \text{const} > 0) : B_\alpha\,(0) = \{x \in \mathbb{R}^n, \parallel x \parallel < \alpha\} \subset D.$$

For every $t \ge t_0$, we introduce the notation

$$\begin{aligned}\triangle = \triangle\,(t; a, \alpha, V_1, V_2, \ldots) &= \bigcap_{i=1,2,\ldots} \triangle_i\,(t; a, \alpha, V_i) \\ &= \bigcap_{i=1,2,\ldots} \{x \in D \colon V_i\,(t,x) \le a\,(\alpha)\}\,.\end{aligned}$$

Under condition 2.5 of the theorem, the sequence of Lyapunov's functions is equipotential continuous at the point 0, $i = 1,2,\ldots$, uniformly in the variable $t \ge t_0$ and therefore the set $\triangle \ne \oslash$. Using the same condition 2.5, we deduce that the set $\triangle$ has a form

$$\triangle = D \cap B_\delta\,(0)\,,$$

where constant $\delta = \delta\,(a, \alpha)$. The point 0 is internal to $\triangle$. Then

$$(\exists \lambda = \text{const} > 0) : B_\lambda\,(0) \subset \triangle.$$

Let t be an arbitrary point, satisfying the inequalities: $t > t_0, t \ne t_i, i = 1,2,\ldots$, and let k be is the biggest number for which the inequality $t > t_k$ is valid. Such a number exists. Indeed, if the switching moments are finite number, this fact is trivial. If the switching moments are infinitely many, then according to Theorem 2 it is fulfilled $\lim_{i\to\infty} t_i = \infty$ and therefore, there exists number k such that $t_k < t < t_{k+1}$. Let $x_0 \in B_\lambda(0) \subset \triangle$. The following inequalities are valid:

$$\begin{aligned}
a(\| x(t;t_0,x_0) \|) &\le V_k\left(t, x\left(t;t_0,x_0\right)\right)\\
&\le V_k\left(t_{k-1}+0, x\left(t_{k-1}+0;t_0,x_0\right)\right)\\
&\le V_k\left(t_{k-1}+0, x\left(t_{k-1};t_0,x_0\right)+I_{k-1}\left(x\left(t_{k-1};t_0,x_0\right)\right)\right)\\
&\le V_{k-1}\left(t_{k-1}-0, x\left(t_{k-1}-0;t_0,x_0\right)\right)\\
&\le V_{k-1}\left(t_{k-2}+0, x\left(t_{k-2}+0;t_0,x_0\right)\right)\\
&\le V_{k-1}\left(t_{k-2}+0, x\left(t_{k-2};t_0,x_0\right)+I_{k-2}\left(x\left(t_{k-2};t_0,x_0\right)\right)\right)\\
&\vdots\\
&\le V_2\left(t_1+0, x\left(t_1+0;t_0,x_0\right)\right)\\
&\le V_2\left(t_1+0, x\left(t_1;t_0,x_0\right)+I_1\left(x\left(t_1;t_0,x_0\right)\right)\right)\\
&\le V_1\left(t_1-0, x\left(t_1-0;t_0,x_0\right)\right)\\
&\le V_1\left(t_0,x_0\right)\\
&\le a\left(\alpha\right).
\end{aligned} \tag{6}$$

Then, from the above inequality, it follows that if the initial point $x_0 \in B_\lambda(0)$, then

$$\| x\left(t;t_0,x_0\right) \| \le \alpha,$$

i.e. the solution of the problem investigated $x\left(t;t_0,x_0\right) \in B_\alpha(0) \subset D$ for $t \ge t_0$. The latter fact means that for every initial point $x_0 \in B_\lambda(0)$ the solution of problem (1), (2), (3), (4) is continued (extended) up to ∞.

The theorem is proved. □

Theorem 5. *Let the conditions of the previous theorem hold.*

Then the zero solution of the system studied (1), (2), (3) is stable in respect to the initial condition.

Proof. Let $\varepsilon > 0$. The following two cases take place:

Case 1. The system considered of differential equations with variable structure and impulses possesses a finite number of switching moments: $t_1, t_2, \ldots, t_k$, $t_0 < t_1 < t_2 < \ldots < t_k$. Let $t_0 \le t < t_1$. Then $x\left(t;t_0,x_0\right)$ is (continuous) solution of the problem without impulses

$$\frac{dx}{dt} = f_1(t,x), \qquad x(t_0) = x_0.$$

Under condition 2.1 of Theorem 4 and using continuity of the function V_1 at point $0 \notin \Phi_1$, it follows that

$$(\forall \varepsilon > 0)(\exists \delta = \delta(t_0,\varepsilon) > 0)(\forall x \in D \cap B_\delta(0)) \Rightarrow V_1(t_0,x) < a(\varepsilon),$$

i.e.

$$\sup\{V_1(t_0,x); \forall x \in D \cap B_\delta(0)\} < a(\varepsilon), \quad \delta = \delta(t_0,\varepsilon) > 0. \tag{7}$$

For $t_0 \le t < t_1$, according to condition 2.4 of Theorem 4, it is satisfied

$$\dot{V}_1(t, x(t;t_0,x_0)) \le 0.$$

Therefore,

$$V_1(t, x(t;t_0,x_0)) \le V_1(t_0,x_0), \qquad t_0 \le t < t_1. \tag{8}$$

Let $x_0 \in D \cap B_\delta(0)$. Then, using consecutively condition 2.2 of Theorem 4, inequality (8) and (7), we find

$$a(\| x(t;t_0,x_0) \|) \le V_1(t, x(t;t_0,x_0)) \le V_1(t_0,x_0) \le a(\varepsilon), \qquad t_0 \le t < t_1. \tag{9}$$

From the inequalities above, using the fact $a \in K$, we conclude that

$$\| x(t;t_0,x_0) \| < \varepsilon, \quad t_0 \le t < t_1.$$

Let $t_1 < t < t_2$. Then the solution of problem (1), (2), (3), (4) is continuous and coincides with the solution of the problem without impulses.

$$\frac{dx}{dt} = f_2(t,x), \qquad x(t_1+0) = x(t_1;t_0,x_0) + I_1(x(t_1;t_0,x_0)).$$

In accordance with condition 2.3 of Theorem 4, it is satisfied

$$\begin{aligned} V_2(t_1+0, x(t_1;t_0,x_0) + I_1(x(t_1;t_0,x_0))) &\le V_1(t_1, x(t_1;t_0,x_0)) \\ &= V_1(t_1-0, x(t_1-0;t_0,x_0)). \end{aligned} \tag{10}$$

Using (9) it is valid

$$V_1(t_1-0, x(t_1-0;t_0,x_0)) \le V_1(t_0,x_0) \le a(\varepsilon). \tag{11}$$

For $t_1 < t < t_2$, according to condition 2.4 of Theorem 4 (similar to (8)) we deduce

$$V_2\left(t, x\left(t; t_0, x_0\right)\right) = V_2\left(t_1 + 0, x\left(t_1 + 0; t_0, x_0\right)\right) \\ \leq V_2\left(t_1 + 0, x\left(t_1; t_0, x_0\right) + I_1\left(x\left(t_1; t_0, x_0\right)\right)\right). \quad (12)$$

where $t_1 < t < t_2$.

Applied consistently condition 2.2 of the previous theorem, inequalities (12), (10) and (11), we obtain

$$\begin{aligned} a\left(\| x\left(t; t_0, x_0\right) \|\right) \leq & V_2\left(t, x\left(t; t_0, x_0\right)\right) \\ \leq & V_2\left(t_1 + 0, x\left(t_1; t_0, x_0\right) + I_1\left(x\left(t_1; t_0, x_0\right)\right)\right) \\ \leq & V_1\left(t_1 - 0, x\left(t_1 - 0; t_0, x_0\right)\right) \\ \leq & V_1\left(t_0, x_0\right) \\ \leq & a\left(\varepsilon\right), \quad t_1 < t < t_2. \end{aligned}$$

Since $a \in K$, then from the last inequalities, it follows

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t_1 < t < t_2.$$

Similar to the inequality above, we obtain consecutively

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t_{i-1} < t < t_i, \; i = 1, 2, \ldots, k, \quad (13)$$

and also

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t_k < t < \infty. \quad (14)$$

Taking into consideration the equations

$$x\left(t_i - 0; t_0, x_0\right) = x\left(t_i; t_0, x_0\right), \quad i = 1, 2, \ldots, \quad (15)$$

then, using (13) and (14), we obtain the conclusion

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t_0 < t < \infty, \quad (16)$$

which completes the proof in this case.

Case 2. There exist infinity many switching moments: $t_1, t_2, \ldots, t_0 < t_1 < t_2 < \ldots$. Similar to the previous case (inequality (13)), we find

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t_{i-1} < t < t_i, i = 1, 2, \ldots.$$

From the inequalities above, using that $\lim_{i \to \infty} t_i = \infty$ (see second part of Theorem 2) and equalities (15) once again, we come to the inequality (16).

The theorem is proved. □

Theorem 6. *Let the following conditions hold:*

1. *The conditions H1–H9 are valid.*
2. *There exists a sequence of scalar piecewise continuous Lyapunov's functions*
$$\left\{V_i; V_i\colon \mathbb{R}^+ \times D \to \mathbb{R}^+,\ i = 1, 2, \ldots\right\},$$
corresponding to the impulsive system differential equations (1), (2), (3), such that:

(2.1) $V_i\,(t, 0) = 0, t \in \mathbb{R}^+,\ i = 1, 2, \ldots;$

(2.2) To the sequence above of Lyapunov's functions
$$\{V_i; i = 1, 2, \ldots\},$$
the functions $a, b \in K$ *correspond, such that*
$$a\,(\|\ x\ \|) \leq V_i\,(t, x) \leq b\,(\|\ x\ \|)\,, \quad (t, x) \in \mathbb{R}^+ \times D,\ i = 1, 2, \ldots;$$

(2.3) It is satisfied
$$V_{i+1}\,(t + 0, x + I_i\,(x)) = V_{i+1}\,(t, x + I_i\,(x)) \leq V_i\,(t, x)\,,$$
$$(t, x) \in \mathbb{R}^+ \times \Phi_i,\ i = 1, 2, \ldots;$$

(2.4) It is satisfied
$$\dot{V}_i\,(t, x) \leq 0, (t, x) \in \mathbb{R}^+ \times (D \diagdown \Phi_i)\,, i = 1, 2, \ldots.$$

Then the impulsive system (1), (2), (3) has uniformly stable zero solution.

Proof. By the inequality
$$V_i\,(t, x) \leq b\,(\|\ x\ \|)\,, (t, x) \in \mathbb{R}^+ \times D, i = 1, 2, \ldots$$
it follows that the Lyapunov's functions sequence $V_1, V_2, \ldots$ is equipotential continuous at point 0, $i = 1, 2, \ldots$, uniformly in argument $t \in \mathbb{R}^+$. It means that the conditions of Theorem 4 are satisfied and therefore, if the initial point x_0 is "sufficiently close" to 0, then the solution $x\,(t; t_0, x_0)$ of the basic problem is defined for $t \in [t_0, \infty)$. For example, let the constant $\lambda > 0$ exists, such that for $(t_0, x_0) \in \mathbb{R}^+ \times B_\lambda\,(0) \subset \mathbb{R}^+ \times D$, the solution is continued up to ∞.

Let $\varepsilon > 0$. We choose the constant $\delta = \delta\,(\varepsilon)\,, 0 < \delta < \lambda$, such that, the inequality $b\,(\delta) < a\,(\varepsilon)$ is valid. Let $x_0 \in B_\delta\,(0)$. Let t be an arbitrary point, satisfying the inequalities: $t > t_0, t \neq t_i, i = 1, 2, \ldots$. Let k be the biggest number, for which the inequality $t > t_k$ is fulfilled. As we noted in the proof of Theorem 2, such a number exists. Analogously to (6) we establish the following inequalities:

$$\begin{aligned}
a\left(\| x\left(t; t_0, x_0\right) \|\right) &\leq V_k\left(t, x\left(t; t_0, x_0\right)\right) \\
&\leq V_k\left(t_{k-1}+0, x\left(t_{k-1}+0; t_0, x_0\right)\right) \\
&\leq V_k\left(t_{k-1}+0, x\left(t_{k-1}; t_0, x_0\right)+I_{k-1}\left(x\left(t_{k-1}; t_0, x_0\right)\right)\right) \\
&\leq V_{k-1}\left(t_{k-1}-0, x\left(t_{k-1}-0; t_0, x_0\right)\right) \\
&\vdots \\
&\leq V_1\left(t_1-0, x\left(t_1-0; t_0, x_0\right)\right) \\
&\leq V_1\left(t_0, x_0\right) \\
&\leq b\left(\| x_0 \|\right) \leq b\left(\delta\right) \leq a\left(\varepsilon\right).
\end{aligned} \tag{17}$$

Therefore, if $x_0 \in B_\delta\left(0\right)$, then

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, \quad t > t_0,\ t \neq t_i,\ i = 1, 2, \ldots.$$

Taking into account the equations

$$x\left(t_i - 0; t_0, x_0\right) = x\left(t_i; t_0, x_0\right), \quad i = 1, 2, \ldots,$$

we reach the conclusion that

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, t > t_0 \geq 0, x_0 \in B_\delta\left(0\right), \delta = \delta\left(\varepsilon\right) > 0.$$

The theorem is proved. □

Theorem 7. *Let the following conditions hold:*

1. *The conditions H1–H9 are valid.*
2. *There exists a sequence of scalar piecewise continuous Lyapunov's functions*

$$\left\{V_i; V_i\colon \mathbb{R}^+ \times D \to \mathbb{R}^+, i = 1, 2, \ldots\right\},$$

corresponding to the impulsive system differential equations (1), (2), (3), such that:

(2.1) $V_i\left(t, 0\right) = 0,\ t \in \mathbb{R}^+,\ i = 1, 2, \ldots;$

(2.2) To the sequence above of Lyapunov's functions

$$\{V_i; i = 1, 2, \ldots\},$$

the functions $a, b \in K$ correspond, such that

$$a(\| x \|) \le V_i(t, x) \le b(\| x \|), (t, x) \in \mathbb{R}^+ \times D, i = 1, 2, \ldots;$$

(2.3) It is satisfied

$$V_{i+1}(t + 0, x + I_i(x)) = V_{i+1}(t, x + I_i(x)) \le V_i(t, x), \quad (t, x) \in \mathbb{R}^+ \times \Phi_i, \ i = 1, 2, \ldots;$$

(2.4) There exists corresponding function $c \in K$, such that

$$\dot{V}_i(t, x) \le -c(\| x \|), \quad (t, x) \in \mathbb{R}^+ \times (D \diagdown \Phi_i), \quad i = 1, 2, \ldots.$$

Then, impulsive system (1), (2), (3) has uniform-asymptotically stable zero solution.

Proof. The proof of the theorem will be split in several parts.

Part 1. Since the conditions of the previous theorem are satisfied, it follows that the zero solution of problem (1) (2) (3) is uniformly stable.

Part 2. As $0 \in D$ and D is a domain, i.e. open set, then it follows that

$$(\exists \alpha = \text{const} > 0) : B_\alpha(0) = \{x \in \mathbb{R}^n, \| x \| < \alpha\} \subset D.$$

For every $t \ge t_0$, as in the proof of Theorem 4, we introduce the notation

$$\begin{aligned} \triangle &= \triangle(t; a, \alpha, V_1, V_2, \ldots) \\ &= \bigcap_{i=1,2,\ldots} \triangle_i(t; a, \alpha, V_i) \\ &= \bigcap_{i=1,2,\ldots} \{x \in D \colon V_i(t, x) \le a(\alpha)\}. \end{aligned}$$

According to the inequalities

$$V_i(t, x) \le b(\| x \|), \quad (t, x) \in \mathbb{R}^+ \times D, \quad i = 1, 2, \ldots$$

the sequence of Lyapunov's functions is equipotential continuous at the point 0, uniformly in the variable $t \in \mathbb{R}^+$ and therefore the set $\triangle \neq \varnothing$. Using condition 2.2 of the theorem, we deduce that

$$\triangle \subset B_\alpha(0) \subset D.$$

Let t be an arbitrary point, satisfying the inequalities: $t > t_0, t \neq t_i, i = 1, 2, \ldots$, and let k be the biggest number for which the inequality $t > t_k$ is valid. Similarly to (17), using the conditions 2.3 and 2.4 of the theorem, we obtain

$$V_k\left(t, x\left(t; t_0, x_0\right)\right) \leq V_1\left(t_0, x_0\right).$$

From the inequality above, it follows that, if the initial point $x_0 \in \triangle$, then

$$V_k\left(t, x\left(t; t_0, x_0\right)\right) \leq a\left(\alpha\right),$$

i.e. the solution of the problem investigated $x\left(t; t_0, x_0\right) \in \triangle \subset B_\alpha\left(0\right) \subset D$ for $t \geq t_0$. The latter fact means that for every initial point $x_0 \in \triangle$, the solution of problem (1), (2), (3), (4) is continuable up to ∞.

Part 3. Let $\varepsilon > 0$. We choose the constant $\eta = \eta\left(\varepsilon\right) > 0$ such that, the inequality $b\left(\eta\right) < a\left(\varepsilon\right)$ is valid. Let the constant T satisfies the inequality

$$T > \frac{b(\alpha)}{c(\eta)}. \tag{18}$$

Assume that for every t, $t_0 \leq t \leq t_0 + T$ the next inequality is valid

$$\| x\left(t; t_0, x_0\right) \| \geq \eta.$$

Then, we obtain successively

$$\begin{aligned}
a\left(\| x\left(t; t_0, x_0\right) \|\right) \leq & V_k\left(t, x\left(t; t_0, x_0\right)\right) \\
\leq & V_k\left(t_k + 0, x\left(t_k + 0; t_0, x_0\right)\right) \\
& + \int_{t_k+0}^{t} \dot{V}_k\left(\tau, x\left(\tau; t_0, x_0\right)\right) d\tau \\
\leq & V_k\left(t_k, x\left(t_k - 0; t_0, x_0\right) + I_{k-1}\left(x\left(t_k - 0; t_0, x_0\right)\right)\right) \\
& - \int_{t_k+0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
\leq & V_{k-1}\left(t_k - 0, x\left(t_k - 0; t_0, x_0\right)\right) \\
& - \int_{t_k+0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
\leq & V_{k-1}\left(t_{k-1} + 0, x\left(t_{k-1} + 0; t_0, x_0\right)\right) \\
& + \int_{t_{k-1}+0}^{t_k-0} \dot{V}_{k-1}\left(\tau, x\left(\tau; t_0, x_0\right)\right) d\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_k+0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
\leq &V_{k-1}\left(t_{k-1}, x\left(t_{k-1}-0; t_0, x_0\right)+I_{k-2}\left(x\left(t_{k-1}-0; t_0, x_0\right)\right)\right) \\
&-\int_{t_{k-1}+0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
\leq &V_{k-2}\left(t_k-0, x\left(t_k-0; t_0, x_0\right)\right) \\
&-\int_{t_{k-1}+0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
&\vdots \\
\leq &V_1\left(t_0, x_0\right)-\int_{t_0}^{t} c\left(\| x\left(\tau; t_0, x_0\right) \|\right) d\tau \\
< &b(\alpha)-c(\eta)\left(t-t_0\right).
\end{aligned}
$$

Then, from the inequalities above, for sufficiently large values of $t, t_0 \leq t \leq t_0+T$ (t - values are “close” to t_0+T), taking into consideration the restriction (18), we obtain the inequality

$$
0 \leq a\left(\| x\left(t; t_0, x_0\right) \|\right) < b(\alpha)-c(\eta)\left(t-t_0\right) < 0,
$$

which is a contradiction. Thereby, the conclusion is that there exists a point $t^*, t_0 < t^* \leq t_0+T$, such that

$$
\| x\left(t^*; t_0, x_0\right) \| < \eta.
$$

Part 4. Let $t \geq t_0+T \geq t^*$. The following inequalities are valid

$$
t > t_k > t_{k-1} > \ldots > t_{k-p} > t^* > t_{k-p-1},
$$

and k is the biggest number with this property. From conditions 2.2, 2.3 and 2.4 of the theorem, we find

$$
\begin{aligned}
a\left(\| x\left(t; t_0, x_0\right) \|\right) &\leq V_k\left(t, x\left(t; t_0, x_0\right)\right) \\
&< V_k\left(t_k+0, x\left(t_k+0; t_0, x_0\right)\right) \\
&= V_k\left(t_k+0, x\left(t_{k-1}-0; t_0, x_0\right)+I_{k-1}\left(x\left(t_{k-1}-0; t_0, x_0\right)\right)\right) \\
&\leq V_{k-1}\left(t_k-0, x\left(t_{k-1}-0; t_0, x_0\right)\right) \\
&\vdots \\
&\leq V_{k-p-1}\left(t_{k-p}-0, x\left(t_{k-p}-0; t_0, x_0\right)\right)
\end{aligned}
$$

$$\leq V_{k-p-1}\left(t^*, x\left(t^*; t_0, x_0\right)\right)$$
$$b(\| x(t^*; t_0, x_0) \|) = b\left(\eta\right) < a\left(\varepsilon\right).$$

Therefore, if $x_0 \in \triangle$, then

$$\| x\left(t; t_0, x_0\right) \| < \varepsilon, t \geq t_0 + T. \tag{19}$$

Part 5. Let $\lambda = \text{const} > 0$ be such that $b\left(\lambda\right) < a\left(\alpha\right)$. Then, if $x_0 \in B_\lambda\left(0\right)$, then it is fulfilled

$$V_1\left(t_0, x_0\right) \leq b\left(\| x_0 \|\right) < b\left(\lambda\right) < a\left(\alpha\right),$$

i.e. $x_0 \in \triangle$. Consequently, if $x_0 \in B_\lambda\left(0\right)$, then the solution $x\left(t; t_0, x_0\right)$ of problem (1), (2), (3), (4) is continuable up to ∞ and the inequality (19) is valid.

The theorem is proved. □

Theorem 8. *Let the following conditions hold:*

1. *The conditions H1–H9 are valid.*
2. *There exists a sequence of scalar piecewise continuous Lyapunov's functions*
$$\left\{V_i; V_i \colon \mathbb{R}^+ \times D \to \mathbb{R}^+, i = 1, 2, \ldots\right\},$$
corresponding to the impulsive system differential equations (1), (2), (3), such that:

(2.1) $V_i\left(t, 0\right) = 0$, $t \in \mathbb{R}^+$, $i = 1, 2, \ldots$;

(2.2) To the sequence above of Lyapunov's functions

$$\left\{V_i; i = 1, 2, \ldots\right\},$$

the functions $a, b \in K$ *correspond, such that*

$$a\left(\| x \|\right) \leq V_i\left(t, x\right) \leq b\left(\| x \|\right), \quad (t, x) \in \mathbb{R}^+ \times D,\ i = 1, 2, \ldots;$$

(2.3) It is satisfied

$$V_{i+1}\left(t + 0, x + I_i\left(x\right)\right) = V_{i+1}\left(t, x + I_i\left(x\right)\right) \leq V_i\left(t, x\right),$$
$$(t, x) \in \mathbb{R}^+ \times \Phi_i,\ i = 1, 2, \ldots;$$

(2.4) There exists a constant $c \in R$, such that

$$\dot{V}_i\left(t,x\right) \leq -c.V_i\left(t,x\right),\left(t,x\right) \in \mathbb{R}^+ \times \left(D \diagdown \Phi_i\right), i = 1,2,\ldots.$$

Then, impulsive system (1), (2), (3) has uniform-asymptotically stable zero solution.

Proof. As in the previous theorem, we show that:

- There exists a constant $\lambda > 0$ such that, if $x_0 \in B_\lambda\left(0\right)$, then the solution $x\left(t;t_0,x_0\right)$ of problem (1), (2), (3), (4) is continuable up to ∞;
- The zero solution of problem (1), (2), (3) is uniformly stable;
- There exists a constant $\alpha > 0$ such that $B_\alpha\left(0\right) \subset D$.

We choose the constant $\lambda > 0$ such that, the inequality $b\left(\lambda\right) < a\left(\alpha\right)$ is valid. Let ε be an arbitrary positive constant and also

$$T = \text{const} > \max\left\{0, \frac{1}{c}\ln\frac{a\left(\alpha\right)}{a\left(\varepsilon\right)}\right\} > 0.$$

Furthermore, we will require the initial point $x_0 \in B_\lambda\left(0\right)$. Let t be an arbitrary point, satisfying the inequalities: $t > t_0 + T, t \neq t_i, i = 1,2,\ldots$. Assume that k is the biggest number, for which the inequality $t > t_k$ is valid. We obtain consecutively

$$\begin{aligned}
V_k\left(t,x\left(t;t_0,x_0\right)\right) \leq & V_k\left(t_k+0,x\left(t_k+0;t_0,x_0\right)\right) \\
& \times \exp\left(-c\left(t-t_k\right)\right) \\
= & V_k\left(t_k,x\left(t_k-0;t_0,x_0\right)+I_{k-1}\left(x\left(t_k-0;t_0,x_0\right)\right)\right) \\
& \times \exp\left(-c\left(t-t_k\right)\right) \\
\leq & V_{k-1}\left(t_k,x\left(t_k-0;t_0,x_0\right)\right)\exp\left(-c\left(t-t_k\right)\right) \\
\leq & V_{k-1}\left(t_{k-1}+0,x\left(t_{k-1}+0;t_0,x_0\right)\right) \\
& \times \exp\left(-c\left(t_k-t_{k-1}\right)\right)\exp\left(-c\left(t-t_k\right)\right) \\
= & V_{k-1}\left(t_{k-1},x\left(t_{k-1}+0;t_0,x_0\right)\right)\exp\left(-c\left(t-t_{k-1}\right)\right) \\
& \vdots \\
\leq & V_1\left(t_0,x\left(t_0+0;t_0,x_0\right)\right)\exp\left(-c\left(t-t_0\right)\right) \\
= & V_1\left(t_0,x_0\right)\exp\left(-c\left(t-t_0\right)\right) \\
\leq & b\left(\parallel x_0 \parallel\right)\exp\left(-c.T\right)
\end{aligned} \tag{20}$$

$$\leq a(\alpha)\exp(-c.T).$$

There are following two cases:

Case 1. The equality $T=0$ is valid. Therefore $\alpha \leq \varepsilon$ and $a(\alpha) \leq a(\varepsilon)$. In this case, using (20) we get

$$a(\| x(t;t_0,x_0) \|) \leq V_k(t, x(t;t_0,x_0)) \leq a(\alpha)\exp(-c.0) \leq a(\varepsilon),$$

i.e.

$$\| x(t;t_0,x_0) \| < \varepsilon, \quad t > t_0 + T,\ t \neq t_i,\ i = 1,2,\ldots.$$

Taking into account the next equalities

$$x(t_i - 0; t_0, x_0) = x(t_i; t_0, x_0), \quad i = 1,2,\ldots,$$

we conclude that

$$\| x(t;t_0,x_0) \| < \varepsilon, \quad t > t_0 + T,\ x_0 \in B_\lambda(0). \tag{21}$$

Case 2. It is fulfilled $T = \frac{1}{c}\ln\frac{a(\alpha)}{a(\varepsilon)}$. Reiterating from (20), we find

$$a(\| x(t;t_0,x_0) \|) \leq a(\alpha)\exp(-c.T) \leq a(\varepsilon),$$

from where similarly to the preceding case, we find inequality (21).

The theorem is proved. □

References

[1] R. Agarwal, F. Karakoc, A survey on oscillation of impulsive delay differential equations, *Computers & Mathematics with Applications*, **60**, No. 6 (2010), 1648-1685.

[2] R. Agarwal, F. Karakoc, Oscillation theory for impulsive partial differential equations, *Communications in Applied Analysis*, **14**, No. 1 (2010), 59-72.

[3] S. Ahmad, G. Stamov, Almost periodic solutions of N-dimensional impulsive competitive systems, *Nonlinear Analysis: Real World Applications*, **10**, No. 3 (2009), 1846-1853.

[4] M. Akhmet, On the smoothness of solutions of impulsive autonomous system, *Nonlinear Analysis*, **60**, No. 2 (2005), 311-324.

[5] M. Akhmet, *Principles of Discontinuous Dynamical Systems*, Springer, New York (2010).

[6] M. Akhmet, M. Beklioglu, T. Ergenc, V. Tkachenko, An impulsive ratio-dependent predator-prey system with diffusion, *Nonlinear Analysis: Real World Applications*, **7**, No. 5 (2006), 1255-1267.

[7] J. Angelova, A. Dishliev, Optimization problems in population dynamics, *Applicable Analysis*, **69** (1999), 207-221.

[8] J. Angelova, A. Dishliev, S. Nenov, I-optimal curve for impulsive Lotka-Volterra predator-prey model, *Computers & Mathematics with Applications*, **43**, No. 10-11 (2002), 1203-1218.

[9] J. Angelova, A. Dishliev, S. Nenov, Optimization problems for impulsive Lotka-Volterra predator-prey model, *International J. of Differential Equations and Applications*, **3**, No. 4 (2000), 7-18.

[10] D. Bainov, A. Dishliev, I. Stamova, Lipschitz quasistability of impulsive differential - difference equations with variable impulsive perturbations, *J. of Computational and Applied Mathematics*, **70**, No. 2 (1996), 267-277.

[11] D. Bainov, S. Nenov, Limit sets of impulsive dynamical systems, In: *Proceedings of the Fourth International Colloquium on Differential Equations* (1994), 31-34.

[12] D. Bainov, V. Petrov, V. Proytcheva, Existence and asymptotic behavior of nonoscillatory solutions of second-order neutral differential equations with "maxima", *J. of Computational and Applied Mathematics*, **83**, No. 2 (1997), 237-249.

[13] D. Bainov, P. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, World Scientific, Singapore (1995).

[14] D. Bainov, P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific & Technical, Harlow (1993).

[15] D. Bainov, P. Simeonov, *Oscillation Theory of Impulsive Differential Equations*, International Publications, FL (1998).

[16] D. Bainov, P. Simeonov, *System with Impulse Effect: Stability Theory and Applications*, Ellis Horwood, Chichester (1989).

[17] M. Benchohra, J. Henderson, S. Ntouyas, Impulsive differential equations and inclusions, *Contemporary Mathematics and its Applications*, Hindawi Publishing Corporations, **2** (2006).

[18] M. Benchohra, J. Henderson, S. Ntouyas, A. Ouahab, Impulsive functional differential equations with variable times and infinite delay, *J. International of Applied Mathematical Sciences*, **2**, No. 1 (2005), 130-148.

[19] M. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Transactions on Automatic Control*, **43**, No. 4 (1998), 475-482.

[20] R. Chukleva, A. Dishliev, K. Dishlieva, Continuous dependence of the solutions of the differential equations with variable structure and impulses in respect of switching functions, *J. International of Applied Science and Technology*, **1**, No. 5 (2011), 46-59.

[21] DeR. Carlo, S. Zak, G. Matthews, Variable structure control of nonlinear multivariable systems: a tutorial, *Proceedings on the IEEE*, **76**, No. 3 (1988), 212-232.

[22] H. Dimov, S. Nenov, Some features of uncontinuable solutions of impulsive dynamical systems, *Extracta Mathematicae*, **11**, No. 3 (1996), 443-456.

[23] A. Dishliev, D. Bainov, Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces, *J. of Math. Analysis and Applications*, **135**, No. 2 (1988), 369-382.

[24] A. Dishliev, D. Bainov, Continuous dependence on initial condition and a parameter of a class of differential equations with variable structure and impulses, *J. International Systems Science*, **22**, No. 4 (1991), 641-658.

[25] A. Dishliev, D. Bainov, Dependence upon initial conditions and parameters of solutions of impulsive differential equations with variable structure, *J. International of Theoretical Physics*, **29**, No. 6 (1990), 655-675.

[26] A. Dishliev, K. Dishlieva, Continuous dependence and stability of solutions of impulsive differential equations on the initial conditions and impulsive moments, *J. International of Pure and Applied Mathematics*, **70**, No. 1 (2011), 39-64.

[27] A. Dishliev, K. Dishlieva, Continuous dependence of the solutions of differential equations under "short" perturbations on the right hand side, *Communications in Applied Analysis*, **2** (2006), 1-13.

[28] A. Dishliev, K. Dishlieva, Orbital Hausdorff continuous dependence of the solutions of impulsive differential equations with respect to impulsive perturbations, *J. International of Pure and Applied Mathematics*, **70**, No. 2 (2011), 167-187.

[29] A. Dishliev, K. Dishlieva, S. Nenov, *Specific Asymptotic Properties of the Solutions of Impulsive Differential Equations. Methods and Applications*, Academic Publications (2011).

[30] K. Dishlieva, Continuous dependence of the solutions of impulsive differential equations on the initial conditions and barrier curves, *Acta Mathematica Scientia* (2011), To Appear.

[31] K. Dishlieva, Differentiability of solutions of impulsive differential equations with respect to the impulsive perturbations, *Nonlinear Analysis Series B: Real World Applications*, **12**, No. 6 (2011), 3541-3551.

[32] A. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic Publishers, Dordrecht (1988).

[33] W. Gao, Hung, J., Variable structure control of nonlinear systems: a new approach, Industrial Electronics, *IEEE Transactions*, **40**, No. 1 (1993), 45-55.

[34] R. Gladilina, A. Icnatyev, Necessary and sufficient stability conditions for invariant sets of nonlinear impulsive systems, *International Applied Mechanics*, **44**, No. 2 (2008), 228-237.

[35] S. Gurgulla, N. Perestyuk, On Lyapunov's second method in systems with impulse effect, *Reports of Acad. Sci. Ukr. SSR, Ser. A*, **10**, 11-14 (1982), In Russian.

[36] S. Hristova, Nonlinear delay integral inequalities for piecewise continuous functions and applications, *J. Ineq. Pure and Applied Mathematics*, **5**, No. 4 (2004), Article 88.

[37] S. Hristova, D. Bainov, Applications of the monotone-iterative technique of V. Lakshmikantham for solving the initial value problem for impulsive differential-difference equations, *Rocky J. Mountain of Mathematics*, **23**, No. 2 (1993), 609-618.

[38] J. Hung, W. Gao, J. Hung, Variable structure control: A survey, *Industrial Electronics, IEEE Transactions*, **40**, No. 1 (1993), 2-22.

[39] A. Ignatyev, On the stability of invariant sets of systems with impulse effect, *Nonlinear Analysis: Theory, Methods & Applications*, **69**, No. 1 (2008), 53-72.

[40] A. Kosseva, S. Kostadinov, K. Schneider, -dichotomy of linear differential equations in arbitrary Banach space, *Chinese Ann. Mathematical Ser. B*, **24**, No. 4 (2003), 485-490.

[41] A. Kosseva, S. Kostadinov, P. Zabrejko, Stability of linear impulsive differential equations with unbounded operator, *Rostock. Mathematical Kolloq.*, **53** (1999), 51-59.

[42] S. Nenov, Impulsive controllability and optimization problems in population dynamics, *Nonlinear Analysis*, **36**, No. 7 (1999), 881-890.

[43] L. Nie, Z. Teng, L. Hu, J. Peng, The dynamics of a Lotka-Volterra predator-prey model with state dependent impulsive harvest for predator, *BioSystems*, **98**, No. 2 (2009), 67-72.

[44] J. Nieto, Rodriges-R. Lopez, Boundary value problems for a class of impulsive functional equations, *Computers & Mathematics with Applications*, **55**, No. 12 (2008), 2715-2731.

[45] X. Mu, F. Tang, Strict Lyapunov functions for impulsive hybrid time-varying systems with discontinuous right-hand side, *J. of Systems Science and Complexity*, **24**, No. 2 (2011), 261-270.

[46] B. Paden, S. Sastry, A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators, *Circuits and Systems, IEEE Transactions*, **34**, No. 1 (1987), 73-82.

[47] V. Plotnikov, N. Kitanov, On Continuous dependence of solutions of impulsive differential inclusions and impulse control problems, *Cybernetic Systems Analysis*, **5** (2002), 71-85, In Russian.

[48] V. Plotnikov, P. Kitanov, Bogolyubov's theorem for continuous dependence of solutions of quasi differential equations with impulses, *Ukrainian J. Mathematical*, **49**, No. 11 (1997), 1504-1511, In Russian.

[49] G. Stamov, I. Stamova, Almost periodic solutions for impulsive neural networks with delay, *Applied Mathematical Modeling*, **31**, No. 7 (2007), 1263-1270.

[50] G. Stamov, I. Stamova, Integral manifolds for impulsive differential-difference equations, *Electronic Modeling*, **4** (2005), 115-120.

[51] G. Stamov, I. Stamova, Second method of Lyapunov and existence of integral manifolds for impulsive differential-difference equations, *J. of Mathematical Analysis and Applications*, **258**, No. 2 (2001), 371-379.

[52] I. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, Walter de Gruyter, Berlin, New York (2009).

[53] L. Wang, L. Chen, J. Nieto, The dynamics of an epidemic model for pest control with impulsive effect, *Nonlinear Analysis: Real World Applications*, **11**, No. 3 (2010), 1374-1386.