International Journal of Differential Equations and Applications
Volume 16 No. 1 2017, 1-10
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v16i1.4582

acadpubl.eu

AN EXISTENCE PRINCIPLE OF OPTIMAL ENERGY CONTROL FOR A DAMPED SANDWICH BEAM

Xuezhang Hou

Mathematics Department
Towson University
Baltimore, Maryland 21252-0001, USA

Abstract: A damped sandwich beam system described by partial differential equations with boundary conditions is investigated in this paper. The system is transferred to an abstract evolution equation interms of the system operators defined on appropriate Hilbert spaces. A variable structural control problem for the system is proposed, and an equivalent control theorem for the system are obtained by means of semigroup of linear operators. Finally, a simulation result is presented.

AMS Subject Classification: 93C20, 35B35
Key Words: variable structural control, beam system, boundary conditions, semigroup of linear operators

1. Introduction

The vibration suppression of flexible structures has been researched extensively in several decades due to its wide range of applications. One widely used technique is to make use of laminated members such as beams which make the compliant middle layer sandwiched between two stiff layers. The advantage of such a structure is to employ the compliant layer create relatively large shear deformation to promote the dissipation of the vibrational energy of the system.

Received: November 10, 2016

The several constrained three-layer sandwich beam models are developed in [1, 2, 3] based on assumptions that the middle layer resists shear but no bending, and the thickness is assumed to be sufficiently small so that the mass may be neglected or included in the outer layers. The outer layers are usually considered as the Euler-Bernoulli beams. For the damped model, it was discussed in [1] that the system can have an analytic semigroup solution, and the optimal damping parameter is also derived in terms of the material parameters of the structure. When the damping is included in the middle layer so that the shear motion are resisted by a force proportional to the rate of shear, it is shown in [4] by the multiplier method that the system associates with an analytic semigroup and vibrational energy is exponentially decay. In this paper, we shall focus on the cantilevered laminated beam as follows

$$\begin{cases} m\omega_{tt}(x,t) + A\omega_{xxxx}(x,t) - B_{\gamma s_x}(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ C_{\gamma s}(x,t) - s_{xx}(x,t) + B\omega_{xxx}(x,t) = 0, & 0 < x < 1, \quad t > 0, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} \omega(0,t) = 0, \quad \omega_x(0,t) = 0, \quad s(0,t) = 0, \\ \omega_{xx}(1,t) = 0, \quad s_x(1,t) = 0, \quad A\omega_{xxx}(1,t) - B_{\gamma s}(1,t) = u(t) \end{cases}$$

where $\omega(x,t)$ stands for the transverse displacement at time t and longitudinal spatial variable x and $s(x,t)$ is the proportion to the shear in the middle layer. The constant $m > 0$ is the density of the beam, $A, B, C > 0$ the stiff constants $\gamma > 0$ the stiffness of the middle layer, and $u(t) \in L^2(0,\infty)$ is the boundary damping control force. The initial conditions prescribed for the system are

$$\omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x). \tag{1.2}$$

The conservative model ($u = 0$) of (1.1) is developed in [1] ,and is shown that the system admits a C_0 -semigroup solution . However, it should be noted that the vibrational energy is a constant in this model. To suppress the vibration, a control must be present in the system. In present paper, the control is imposed at the boundary $x = 1$ due to its easy implementation in engineering practice. For mathematical modelling process and the other physical background of the system, we refer to [1] for more details.

Suppose the output of the system (1.1) is $y(t) = \omega_t(1,t)$. We propose the boundary output feedback control $u(t) = k_y(t)$ where k is the positive constant feedback gain. Then the boundary conditions of (1.1) become

$$\begin{cases} \omega(0,t) = 0, \quad \omega_x(0,t) = 0, \quad s(0,t) = 0, \\ \omega_{xx}(1,t) = 0, \; s_x(1,t) = 0, \; A\omega_{xxx}(1,t) - B_{\gamma s}(1,t) = k\omega_t(1,t). \end{cases} \tag{1.3}$$

Let us introduce a second order differential operator $\mathcal{T}$ by ([1])

$$\begin{cases} \mathcal{T}_{\varphi} = \varphi'', \\ \mathcal{D}(\mathcal{T}) = \{\varphi \in H^2(0,1) | \varphi(0) = \varphi'(1) = 0\}. \end{cases} \tag{1.4}$$

One can check that $\mathcal{T}$ is densely defined and negative definite in $L^2(0,1)$. Set $\alpha := C_\gamma > 0$ .Then, it is easily verified that $(\alpha - \mathcal{T})^{-1}$ exists and is compact on $L^2(0,1)$. Now, let

$$\mathcal{J} = -I + \alpha(\alpha - \mathcal{T})^{-1} \tag{1.5}$$

where I is the identity operator on $L^2(0,1)$. Obviously, $\mathcal{J}$ is a non-positive bounded operator on $L^2(0,1)$ and

$$\mathcal{J}_{\varphi} = (\alpha - \mathcal{T})^{-1}\mathcal{T}_{\varphi}, \quad \forall \varphi \in D(\mathcal{T}).$$

With the operator $\mathcal{J}$ at hand, the closed-loop laminated beam system (1.1) – (1.3) can be rewritten as $s(x,t) = -B(\alpha - \mathcal{T})^{-1}\omega_{xxx}(x,t)$ with ω satisfying

$$\begin{cases} m\omega_{tt}(x,t) + A\omega_{xxxx}(x,t) + B_\gamma^2(\mathcal{J}\omega_x)_x(x,t) = 0, \\ \omega(0,t) = \omega_x(0,t) = \omega_{xx}(1,t) = 0, \\ A\omega_{xxx}(1,t) + B_\gamma^2 \mathcal{J}\omega_x(1,t) = k\omega_t(1,t). \end{cases} \tag{1.6}$$

The total energy of the system (1.6) is given by

$$E(t) = \frac{1}{2}\int_0^1 m\omega_t^2(x,t) + A\omega_{xx}^2(x,t) - [B_\gamma^2 \mathcal{J}\omega_x(x,t)]\omega_x(x,t)\mathrm{d}x. \tag{1.7}$$

2. Semigroup Properties and Well-Posedness of the System

We begin by formulating the problem (1.6) on the energy state Hilbert space $\mathcal{H}$:

$$\mathcal{H} = H_\omega^2(0,1) \times L^2(0,1), \quad H_\omega^2(0,1) = \{\varphi \in H^2(0,1) | \varphi(0) = \varphi'(0) = 0\}. \tag{2.1}$$

where and henceforth the primes above symbols representing functions denote differentiation with respect to spatial variable x. Due to energy function (1.7) , it is natural to define the following inner product induced norm $\|\cdot\|$ on $\mathcal{H}$ as

$$\|(\omega, z)\|^2 := \int_0^1 m \mid z(x) \mid^2 + A \mid \omega''(x) \mid^2 - B_\gamma^2(\mathcal{J}\omega'(x)\overline{\omega'(x)}\mathrm{d}x, \ \forall (\omega, z) \in \mathcal{H} \tag{2.2}$$

which makes sense because $\mathcal{J}$ is negative on $L^2(0,1)$. Next, define a linear operator $\mathcal{A}:\mathcal{D}(\mathcal{A})(\subset\mathcal{H})\to\mathcal{H}$ by

$$\mathcal{A}(\omega,z)=\left(z,-\frac{1}{m}[A\omega'''+B_\gamma^2(\mathcal{J}\omega')]'\right),\quad \forall(\omega,z)\in\mathcal{D}(\mathcal{A}) \tag{2.3}$$

where

$$\mathcal{D}(\mathcal{A})=\left\{(\omega,z)\in\mathcal{H}\,\middle|\,\begin{array}{l}\omega'\in\mathcal{D}(\mathcal{T}),\ z\in H_\omega^2(0,1),\ A\omega'''+B_\gamma^2(\mathcal{J}\omega')\in H^1(0,1),\\ \omega''(1)=0,\ A\omega'''(1)+B_\gamma^2(\mathcal{J}\omega')(1)=kz(1)\end{array}\right\}. \tag{2.4}$$

Set $Y(t)=(\omega(\cdot,t),\ \omega_t(\cdot,t))$. Then the system (1.6) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}Y(t)=\mathcal{A}Y(t), & t>0,\\ Y(0)=(\omega(\cdot,0),\ \ \omega_t(\cdot,0))\end{cases} \tag{2.5}$$

Theorem 2.1. *Let $\mathcal{A}$ be the operator defined by* (2.3) *and* (2.4). *Then $\mathcal{A}$ is dissipative in $\mathcal{H}$. In addition, $\mathcal{A}^{-1}$ exist and is compact on $\mathcal{H}$. Therefore, $\mathcal{A}$ generates a C_o-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$ and the spectrum.*

Proof. Since $\mathcal{J}$ is negative on L^2 and

$$\begin{aligned}\langle\mathcal{A}(\omega,z),\ (\omega,z)\rangle_{\mathcal{H}}=&\left\langle\left(z,-\frac{1}{m}[A\omega'''+B_\gamma^2(\mathcal{J}\omega')]'\right),(\omega,z)\right\rangle_{\mathcal{H}}\\ =&-\left\langle[A\omega'''+B_\gamma^2(\mathcal{J}\omega')]',z\right\rangle_{L^2}+A\langle z'',\omega''\rangle_{L^2}-B_\gamma^2\langle\mathcal{J}z',\omega'\rangle_{L^2}\\ =&-[A\omega'''(x)+B_\gamma^2(\mathcal{J}\omega')(x)]\overline{z(x)}|_0^1+A\omega''(x)\overline{z'(x)}|_0^1\\ &-\langle A\omega'',z''\rangle_{L^2}\\ &+B_\gamma^2\langle\mathcal{J}\omega',z'\rangle_{L^2}+A\langle z'',\omega''\rangle_{L^2}-B_\gamma^2\langle\mathcal{J}z',\omega'\rangle_{L^2}\\ =&-k|z(1)|^2-\langle A\omega'',z''\rangle_{L^2}+B_\gamma^2\langle\mathcal{J}\omega',z'\rangle_{L^2}+A\langle z'',\omega''\rangle_{L^2}\\ &-B_\gamma^2\langle\mathcal{J}z',\omega'\rangle_{L^2},\end{aligned}$$

it follows that

$$\mathrm{Re}\langle\mathcal{A}(\omega,z),\ (\omega,z)\rangle_{\mathcal{H}}=-k|z(1)|^2\le0.$$

Hence $\mathcal{A}$ is dissipative. We accomplish the proof by showing that $0\in\rho(\mathcal{A})$ because from Theorem 4.6 of [5], if $\mathcal{A}^{-1}$ exists, $\mathcal{A}$ must be densely defined in

$\mathcal{H}$. Therefore, the Lumer-Phillips can be applied to conclude that $\mathcal{A}$ generates a C_o-semigroup of contractions on $\mathcal{H}$.

To do so, given $G = (g_1, g_2) \in \mathcal{H}$, we seek $F = (f_1, f_2) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}F = G_1$$

from which we obtain that $f_2(x) = g_1(x)$ with f_1 satisfying

$$[Af_1''' + B_\gamma^2(\mathcal{J}f_1')]'(x) = -mg_2(x). \tag{2.6}$$

The equation above is equivalent to the following equation

$$[A + B_\gamma^2(\alpha - \mathcal{T})^{-1}]f_1'''(x) = -m_2\int_1^x g_2(\tau)\mathrm{d}\tau + kg_1(1) =: \phi(x) \in L^2(0,1).$$

It follows that

$$f_1'''(x) = [A^{-1} - B_\gamma^2(\alpha + A^{-1}B_\gamma^2 - \mathcal{T})^{-1}]\phi(x) := \varphi(x) \in L^2(0,1).$$

Solving the equation above yiels

$$f_1(x) = \int_1^x \frac{(x-\tau)^2}{2}\psi(\tau)\mathrm{d}\tau - x\int_0^1 \tau\psi(\tau)\mathrm{d}\tau + \int_0^1 \frac{\tau^2}{2}\psi(\tau)\mathrm{d}\tau.$$

Hence, there is a unique solution $f_1(x)$ to (2.6) , and therefore we can conclude that $\mathcal{A}^{-1}$ exists. Consequently, applying the Sobolev embedding theorem, we see from above expression that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$ and hence the spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues only [6].

3. The Existence Principle of an Optimal Energy Control

Let us establish a sliding model control for the system (2.5)as follows

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \mathcal{A}Y(t) + Bw(Y,t), \quad t > 0, \\ Y(0) = Y_0 = (\omega(\cdot,0), \ \ \omega_t(\cdot,0)) \end{cases} \tag{3.1}$$

where B is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$, $w(Y,t)$ is the control of the system (3.1) that is not continuous on the manifold $S = CY = 0$, and C is a bounded linear operator with $S = S(Y) = CY \in R^n$.

Now, we consider the δ-neighborhood of sliding mode $S = CY = 0$, where $\delta > 0$ is an arbitrary given positive number. Using a continuous control $\tilde{w}(Y,t)$ instead of $w(Y,t)$ in the system (3.1) yields

$$\begin{cases} \dot{Y} = \mathcal{A}Y + B\tilde{w}(Y,t) \\ z(0) = z_0 \end{cases} \tag{3.2}$$

where $\dot{Y} = \partial Y/\partial t$, and the solution of (3.2) belongs to the boundary layer $\|S(Y)\| \leq \delta$

Let $\dot{S}(Y) = C\dot{Y} = 0$. Applying C to the first equation of (3.1) leads to the following the equivalent control:

$$w_{eq}(Y,t) = -(CB)^{-1}C(\mathcal{A}Y)$$

with assumption that $(CB)^{-1}$ exists. Substitute $w_{eq}(Y,t)$ into (3.1) to find

$$\begin{cases} \dot{z} = [I - B(CB)^{-1}C]\mathcal{A}Y. \end{cases} \tag{3.3}$$

Denote $P = B(CB)^{-1}C$ and $\mathcal{A}_0 = (I - P)\mathcal{A}$, then (3.1) becomes

$$\begin{cases} \dot{Y} = \mathcal{A}_0 Y, \qquad Y(0) = Y_0 \end{cases} \tag{3.4}$$

In the rest part of this paper, we are going to show that the actual sliding mode $Y(t)$ will approach uniformly to the ideal sliding mode $\overline{Y}(t)$ under certain conditions.

Lemma 3.1. *If $(CB)^{-1}$ is a compact operator and $P\mathcal{A} = \mathcal{A}P$, then $\mathcal{A}_0 = (I - P)\mathcal{A}$ generates a C_0-semigroup $T_2(t)$ in $\mathcal{H}$ and $T_2(t) = (I - p)T_1(t)$, where $T_1(t)$ is the C_0-semigroup generated by $\mathcal{A}$.*

Proof. Since $(CB)^{-1}$ is a compact operator, B and C are bounded linear operators, we see from the definition of P that P is compact, and therefor the range of $I - P$ is a closed subspace of $\mathcal{H}$. Since $P^2 = P$ and $(1 - P)^2 = I - P$, $I - P$ can be viewed as the identity operator on $(I - P)\mathcal{H}$. It can be easily seen that $T_2(t) = (I - P)T_1(t)$ is a C_0-semigroup in $(I - P)\mathcal{H}$.

Next, we shall prove that the infinitesimal generator of $T_2(t)$ is $(I - P)\mathcal{A}$ and $\mathcal{D}((I - P)\mathcal{A}) = (I - P)\mathcal{D}(\mathcal{A})$.

In fact, for every $x \in (I - P)\mathcal{D}(\mathcal{A})$, there is a $x_1 \in \mathcal{D}(\mathcal{A})$ such that $x = (I - P)x_1$. It should be noted that $T_1(t)$ and $I - P$ are commutative because $\mathcal{A}$ and P are commutative. We see that

$$\begin{aligned} \lim_{t \to 0^+} \frac{T_2(t)x - x}{t} &= \lim_{t \to 0^+} \frac{(I - P)T_1(t)(I - P)x_1 - (I - P)x_1}{t} \\ &= \lim_{t \to 0^+} \frac{(I - P)^2 T_1(t)x_1 - (I - P)x_1}{t} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0^+} \frac{(I-P)T_1(t)x_1 - (I-P)x_1}{t} \\
&= (I-P)\lim_{t\to 0^+} \frac{T_1(t)x_1 - x_1}{t} \\
&= (I-P)\mathcal{A}x_1.
\end{aligned}
$$

Let $\tilde{\mathcal{A}}$ be the infinitesimal generator of $T_2(t)$. Since the limit on the left exists, we can assert that $x \in \mathcal{D}(\tilde{\mathcal{A}})$ and $(I-P)\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\tilde{\mathcal{A}})$.

On the other hand, for any $x \in \mathcal{D}(\tilde{\mathcal{A}})$, since $\mathcal{D}(\tilde{\mathcal{A}}) \subseteq (I-P)\mathcal{H}$, there exists $\tilde{x} \in \mathcal{H}$, such that $x = (I-P)\tilde{x}$, and

$$
\begin{aligned}
\lim_{t\to 0^+} \frac{T_2(t)x - x}{t} &= \lim_{t\to 0^+} \frac{T_2(t)(I-P)\tilde{(x)} - (I-P)\tilde{(x)}}{t} \\
&= \lim_{t\to 0^+} \frac{(I-P)T_1(t)\tilde{x} - (I-P)\tilde{x}}{t} \\
&= (I-P)\lim_{t\to 0^+} \frac{T_1(t)\tilde{x} - \tilde{x}}{t} \\
&= (I-P)\mathcal{A}\tilde{x}.
\end{aligned}
$$

Since the limit of the left hand side exists, and so the limit of the right hand side exists, and $\tilde{x} \in \mathcal{D}(\mathcal{A})$ which implies that $\mathcal{D}(\tilde{\mathcal{A}}) \subseteq (I-P)\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{D}(\tilde{\mathcal{A}}) = (I-P)\mathcal{D}(\mathcal{A})$ and $\tilde{\mathcal{A}}$, the infinitesimal generator of $T_2(t)$, is $(I-P)\mathcal{A}$.

The proof of the lemma is complete. □

Theorem 3.2. *Suppose that the system (3.1) satisfies*

1. *$(CB)^{-1}$ exists and it is compact,*
2. *$P\mathcal{A} = \mathcal{A}P$, where $P = B(CB)^{-1}C$.*

Then for any solution $Y(t)$ of the system (3.4) satisfying $S(\overline{Y}_0) = 0$, $\overline{Y}_0 \in \mathcal{D}(\mathcal{A}_0)$ and $\|Y_0 - \overline{z}_0\| \le \delta$, $Y_0 \in \mathcal{D}(\mathcal{A})$, we have

$$
\lim_{\delta\to 0} \|Y(t) - \overline{Y}(t)\| = 0
$$

uniformly on $[0, T]$ for any positive number T.

Proof. We see from the Theorem 2.1 and Lemma (3.1) that $\mathcal{A}$ and $\mathcal{A}_0 = (I-P)\mathcal{A}$ are infinitesimal generators of C_0-semigroups $T_1(t)$ and $T_2(t)$ respectively. It follows from theory of semigroup of linear operators that there are positive constants $M_1,\ M_2,\ \omega_1$ and ω_2 such that

$$\begin{cases}\|T_1(t)\| \leq M_1 e^{\omega_1 t}, \quad \|T_2(t)\| \leq M_2 e^{\omega_2 t}. \quad (0 \leq t \leq T)\end{cases} \tag{3.5}$$

In the boundary layer $\|T_1(t)\| \leq \delta$, the equivalent control is

$$\begin{cases}w_{eq}(Y,t) = -(CB)^{-1}C\mathcal{A}Y + (CB)^{-1}C\dot{Y}\end{cases}$$

Substitute (3.6) into (3.1) to find

$$\begin{cases}\dot{Y} = (I-P)\mathcal{A}Y + P\dot{Y}\end{cases}$$

Hence, the solution of (3.1) can be expressed as follows:

$$\begin{cases}Y(t) = T_2(t)Y_0 + \int_0^t T_2(t-s)P\dot{Y}(s)ds,\end{cases} \tag{3.6}$$

and the solution of (3.4) can be written as

$$\begin{cases}\overline{Y}(t) = T_2(t)\overline{Y}_0\end{cases} \tag{3.7}$$

Substracting (3.7) from (3.6) yields

$$\begin{cases}Y(t) - \overline{Y}(t) = T_2(t)(Y_0 - \overline{Y}_0) + \int_0^t T_2(t-s)P\dot{Y}(s)ds\end{cases} \tag{3.8}$$

Since $P\mathcal{A} = \mathcal{A}P$, we see that $PT_1(t) = PT_1(t)$. It should be emphasized that $(I-P)P = 0$ and $T_2(t) = (I-P)T_1(t)$, and consequently,

$$\begin{aligned}\int_0^t T_2(t-s)P\dot{Y}(s)ds &= \int_0^t (I-P)T_1(t-s)P\dot{Y}(s)ds \\ &= \int_0^t T_1(t-s)(I-P)P\dot{Y}(s)ds \\ &= 0\end{aligned}$$

It can be obtained from (3.8) and (3.5) that

$$\|z(t) - \overline{Y}(t)\| \leq \|T_2(t)\|\|Y_0 - \overline{z}_0\| \leq M_2 e^{\omega_2 T}\|Y_0 - \overline{Y}\|,$$

Since $\|Y_0 - \overline{Y}_0\| \leq \delta$, we have

$$\|Y(t) - \overline{Y}(t)\| \leq M_2 e^{\omega_2 T}\delta.$$

Thus,

$$\lim_{\delta \to 0}\|Y(t) - \overline{Y}_0\| = 0.$$

The proof of the theorem is complete. □

We see from the Theorem 3.2 that the actual sliding mode can be approximated by ideal sliding mode in any accuracy.

References

[1] R. H. Fabiano and S. W. Hansen, Modeling and analysis of a three-layerdamped sandwich beam, *Discrete Contin. Dynam. Syst.*, Addea Volume (2001), 143-155.

[2] S. W Hansen and R. Spies, Structural damping in alaminated beam due to interfacial slip, emphJ. Sound Vivration, 204(1997), 183-202.

[3] J. M. Wang, B. Z Guo and B. Chentouf, Boundary Feedback Stabilization of Three-Layer Sandwich Beam: Riesz Basis Approach, *ESAIM: Control, Optimization and Calculus of Variations*, 12(2006), 12-34.

[4] S. W Hansen and I. Laciecla, Analyticity, hyperbolicity and uniform stability of semigroupsm arising in models of composite beams, emphMath. Models Methods Appl. Sci., 10(2000), 555-580.

[5] A. Pazy, Semigroups of linear operators and applic ations to partial differential equations.*Spring-Verlag,Berlin*(1983).

[6] T. Cato, Perturbation theory of linear operators, emphSpringer, Berlin, 1976.

[7] G. Chen and J. Zhou, Vibration and damping in distributed systems, Vols.I,II. *Stud. Adv. Math.* (1993).

[8] F. Conrad and O. Mörgul, On the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim,* 36 (1998), 1962-1986.

[9] B. Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim,* 39 (2001), 1736-1747.

[10] B.Z. Guo, Y. Xie and Xuezhang Hou, "On Spectrum of a General Petrowsky Type Equation and Riesz Basis of N-Connected Beam with Linear Feedback at Joints", *Journal of Dynamical and Control Systems,* Vol.10,No.2 (2004), 187-211.

[11] B. Z. Guo and Y. H. Luo, Riesz basis property of a second order hyperbolic system with collocated scalar input/output. *IEEE Trans. Automat. Control* 47(2002), 693-698.

[12] L. F. Ho, Controllability and stabilizability of coupled strings with control applied at the coupled points. *SIAM J. Control Optim,* 31 (1993), 1416-1437.

[13] R. F. Curtain and H. J. Zwart, An introduction to infinite-dimensional linear system theory. *Springer-Verlag, New York* (1995).

[14] K. S. Liu, Energy decay problems in the design of a point stabilizer for connected string vibrating systems. *SIAM J. Control Optim,* 26 (1988), 1348-1356.

[15] K. S. Liu, F. L. Huang, and G. Chen, Exponential stability analysis of a long chain of connected vibrating strings with dissipative linkage. *SIAM J. Appl. Math.* 49 (1989), 1694-1707.

[16] Z. H. Luo, and B. Z. Guo, Shear force feedback control of a single-link flexible robot with a revolute joint. *IEEE Trans. Automat. Control* 42 (1997), 53-65.

[17] Z. H. Luo, B. Z. Guo, and O. Mörgul, Stability and stabilization of infinite-dimensional system with applications. *Spring-Verlag, London* (1999).

[18] S.G. Krantz and W. H. Paulsen, Asymptotic eigenfrequency distributions for the N-beam Euler-Bernoulli connected beam equation with dissipative joints. *J. Symbolic Comput.* 11 (1991), 369-418.

[19] R. Rebarber, Exponential stability of connected beams with dissipative joints: a frequency domain approach. *SIAM J. Control Optim,* 33 (1995), No.I, 1-28.