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ASYMPTOTIC THEORY FOR A CRITICAL CASE FOR A GENERAL THIRD-ORDER DIFFERENTIAL EQUATION

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Abstract: In this paper we identify a relation between the coefficients that represent a critical case for a general third-order differential equation. We obtain the asymptotic form of solutions for this critical case.

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1. Introduction

In this paper we examine the asymptotic form of a fundamental set of solutions of the general third-order differential equation

$$\{q(qy')'\}' + \frac{1}{2}\{(q_1y)' + q_1y'\} + (p_0y')' + p_1y = 0, \tag{1}$$

as $x \to \infty$, where x is the independant variable and the prime denotes $\frac{d}{dx}$. The functions q, q_1, p_0 and p_1 are defined on an interval $[a, \infty)$ and are all nowhere zero there and are not necessarily real-valued.

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In this paper, we consider the case where qq_1 and p_0p_1 are dominated by p_0^2 and q_1^2 consecutively for large x. Under this case we identify the following critical case:

$$\frac{q_1'}{q_1} \to \sigma_1 \text{ and } \frac{(qp_0^{-2})'}{qp_0^{-2}} \to \omega_1, \quad (x \to \infty) \tag{2}$$

where σ_1 and ω_1 are nonzero constants. The critical case (2) is given in (66) and (67). As we shall see in example 1 in Section 5, (2) is given by

$$\alpha_1 - \alpha_4 = 1. \tag{3}$$

Similar third and fourth-order equations to (1) have also been investigated by Al-Hammadi [1], [2], Pfeiffer [9] and Unsworth [10]. In this paper we use the recent theorem of Eastham (see Section 2 of [3], [5]) to obtain the solutions of (1). The general features of our method are given in Section 2 and 3, with main theorem for (1) in Section 4. Finally, in Section 5, we give some examples.

2. The System $Z' = (\Lambda - T^{-1}T')Z$

We write (1) in the standard way [7] as a first-order system :

$$Y' = AY, \tag{4}$$

where the first component of Y is y and

$$A = \begin{pmatrix} 0 & q^{-1} & 0 \\ \frac{-1}{2}q_1q^{-1} & -p_0q^{-2} & q^{-1} \\ -p_1 & \frac{-1}{2}q_1q^{-1} & 0 \end{pmatrix}. \tag{5}$$

As in [1], we express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{6}$$

and we therefore require the eigenvalues λ_j and eigenvectors v_j $(1 \le j \le 3)$ of A.

Let us set

$$q^2 = q_0. \tag{7}$$

The characterestic equation of A is given by

$$q_0\lambda^3 + p_0\lambda^2 + q_1\lambda + p_1 = 0. \tag{8}$$

At this stage, we require the following conditions in the coefficients q_0, p_0, q_1 and p_1 as $x \to \infty$ as follows:

(i) q_0, p_0, q_1 and p_1 are nowhere zero in some interval $[a, \infty)$ and

$$q_0 q_1 = o(p_0^2) \ \ (x \to \infty), \tag{9}$$

$$p_0 p_1 = o(q_1^2) \ \ (x \to \infty), \tag{10}$$

and we are in position to conclude

$$\epsilon_1 = \frac{q_0 q_1}{p_0^2} \to 0 \ \ (x \to \infty), \tag{11}$$

$$\epsilon_2 = \frac{p_0 p_1}{q_1^2} \to 0 \ \ (x \to \infty), \tag{12}$$

$$\epsilon_3 = \frac{q_0 p_1}{q_1 p_0} \to 0 \ \ (x \to \infty), \tag{13}$$

As in [1], we can solve the characterestic equation (8) subject to (9) and 10. Then (8) gives the distinct eigenvalues λ_j as

$$\lambda_1 = -\frac{p_1}{q_1}(1 + \delta_1), \tag{14}$$

$$\lambda_2 = -\frac{q_1}{p_0}(1 + \delta_2), \tag{15}$$

and

$$\lambda_3 = -\frac{p_0}{q_0}(1 + \delta_3), \tag{16}$$

where

$$\delta_1 = O(\epsilon_2), \tag{17}$$

$$\delta_2 = O(\epsilon_1) + O(\epsilon_2), \tag{18}$$

and

$$\delta_3 = O(\epsilon_1). \tag{19}$$

Thus again by (9) and (10), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_{j+1}) \ \ (x \to \infty, 1 \le j \le 2). \tag{20}$$

As we shall see in Section 4, because of different orders of the eigenvalues as $x \to \infty$, the solutions of (1) have different orders of magnitude as $x \to \infty$.
An eigenvector v_j of A corresponding to λ_j is

$$v_j = (1, q_0^{1/2}\lambda_j, 1/2q_1 + p_0\lambda_j + q_0\lambda_j^2)^t, \tag{21}$$

where superscript t denotes transpose. We define the matrix T in (6) by

$$T = (v_1 v_2 v_3). \tag{22}$$

Now as in [1] from (5) we note that EA coincides with its own transpose, where

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{23}$$

Hence by [4] in Section 2(i), the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \;\; (k \neq j). \tag{24}$$

As in [1], we define the scalars m_j $(1 \leq j \leq 3)$ by

$$m_j = (Ev_j)^t v_j, \tag{25}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{26}$$

Then by [4], Section 2,

$$T^{-1} = \begin{bmatrix} m_1^{-1} r_1 \\ m_2^{-1} r_2 \\ m_3^{-1} r_3 \end{bmatrix}. \tag{27}$$

and the

$$m_j = 3q_0\lambda_j^2 + 2p_0\lambda_j + q_1. \tag{28}$$

Now by (6), the transformation

$$Y = TZ \tag{29}$$

takes (4) into

$$Z' = (\Lambda - T^{-1}T')Z, \tag{30}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \tag{31}$$

Now since (21)-(27) all hold, it follows directly from Section 5 in chapter 3 of [5], that the matrix $T^{-1}T' = (t_{jk})$ is given by

$$t_{jj} = \frac{1}{2}\frac{m_j'}{m_j'}, \tag{32}$$

and, for $j \neq k$

$$t_{jk} = (\lambda_j - \lambda_k)^{-1} m'_j \left\{ \frac{1}{2}(\lambda_j + \lambda_k)(q'_0\lambda_j\lambda_k + q'_1) + (p'_0\lambda_j\lambda_k + p'_1) \right\}. \tag{33}$$

Now we need to work out (32) and (33) in terms of q_0, p_0, q_1 and p_1 to determine the form (30) and then make progress towards (1).

3. The Matrix $T^{-1}T'$

In our analysis, we require the following condition on the derivative of q_0, p_0, q_1 and p_1

$$(ii) \qquad \frac{q'_0}{q_0}\epsilon_1,\ \frac{q'_0}{q_0}\epsilon_2,\ \frac{q'_1}{q_1}\epsilon_1,\ \frac{q'_1}{q_1}\epsilon_2,\ \frac{p'_0}{p_0}\epsilon_1,\ \frac{p'_0}{p_0}\epsilon_2,\ \frac{p'_1}{p_1}\epsilon_2,\ \frac{p'_1}{p_1}\epsilon_3, \tag{34}$$

are all in $L(a, \infty)$.

In (32), we require the form of the m_j and m'_j as $x \to \infty$, First by (14)-(19), (28) gives

$$m_1 = q_1(1 + O(\epsilon_2)), \tag{35}$$

$$m_2 = -q_1(1 + O(\epsilon_1) + O(\epsilon_2)), \tag{36}$$

and

$$m_3 = \frac{p_0^2}{q_0}(1 + O(\epsilon_1)). \tag{37}$$

Also on substituting (14), (15) and (16) into (28) for $j = 1, 2, 3$ and differentiating, we obtain

$$m'_1 = q'_1 \{1 + O(\epsilon_2)\} + q_1 \left\{O(\epsilon'_2) + O(\epsilon_2\delta'_1) + O(\epsilon_2\epsilon'_3)\right\} \tag{38}$$

$$m'_2 = q'_1 \{1 + O(\epsilon_1) + O(\epsilon_2)\} + q_1 \left\{O(\delta'_2) + O(\epsilon'_2)\right\}, \tag{39}$$

and

$$m'_3 = \big(2\frac{p'_0}{p_0} - \frac{q'_0}{q_0}\big)\frac{p_0^2}{q_0} \{1 + O(\epsilon_1)\} + \frac{p_0^2}{q_0} \left\{O(\delta'_3) + O(\epsilon'_1)\right\}. \tag{40}$$

Further, by (11)-(13),

$$\epsilon'_1 = O\big(\frac{q'_0}{q_0}\epsilon_1\big) + O\big(\frac{q'_1}{q_1}\epsilon_1\big) + O\big(\frac{p'_0}{p_0}\epsilon_1\big), \tag{41}$$

$$\epsilon'_2 = O\big(\frac{p'_0}{p_0}\epsilon_2\big) + O\big(\frac{p'_1}{p_1}\epsilon_2\big) + O\big(\frac{q'_1}{q_1}\epsilon_2\big), \tag{42}$$

and

$$\epsilon_3' = O\big(\frac{q_0'}{q_0}\epsilon_3\big) + O\big(\frac{q_1'}{q_1}\epsilon_3\big) + O\big(\frac{p_0'}{p_0}\epsilon_3\big) + O\big(\frac{p_1'}{p_1}\epsilon_3\big). \tag{43}$$

Then for reference shortly, we note on substituting (14), (15) and (16) into (8) and differentiating, we obtain

$$\delta_1' = O(\epsilon_2') + O(\epsilon_2\epsilon_3') \tag{44}$$

$$\delta_2' = O(\epsilon_1') + O(\epsilon_2') \tag{45}$$

and

$$\delta_3' = O(\epsilon_1') + O(\epsilon_3') \tag{46}$$

Hence by (41)- (46) and (34)

$$\epsilon_j' \text{ and } \delta_j' \text{ are } L(a,\infty),\ (1 \le j \le 3). \tag{47}$$

For the diagonal elements t_{jj} $(1 \le j \le 3)$, we can now substitute the estimates (35)- (40) into (32). We obtain

$$t_{11} = \frac{1}{2}\frac{q_1'}{q_1} + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O(\epsilon_2') + O(\epsilon_2\delta_1') + O(\epsilon_2\epsilon_3'), \tag{48}$$

$$t_{22} = \frac{1}{2}\frac{q_1'}{q_1} + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O(\delta_2') + O(\epsilon_1'), \tag{49}$$

and

$$t_{33} = 2\frac{p_0'}{p_0} - \frac{q_0'}{q_0} + O\big(\frac{p_0'}{p_0}\epsilon_1\big) + O\big(\frac{q_0'}{q_0}\epsilon_1\big) + O(\delta_3') + O(\epsilon_1'). \tag{50}$$

Now, for the non diagonal elements t_{jk} $(j \ne k,\ 1 \le j,k \le 3)$, we consider (33). Now, by (14), (16), (17), (19), and (35)

$$\frac{(\lambda_1+\lambda_3)(q_0'\lambda_1\lambda_3 + q_1')}{2(\lambda_1-\lambda_3)m_1} = -\frac{1}{2}\frac{q_0'}{q_0} + O\big(\frac{q_0'}{q_0}\epsilon_2\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O\big(\frac{q_1'}{q_1}\epsilon_3\big), \tag{51}$$

and

$$\frac{(p_0'\lambda_1\lambda_3 + p_1')}{(\lambda_1-\lambda_3)m_1} = O\big(\frac{p_0'}{p_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_3\big). \tag{52}$$

Hence by (45) and (52), (33) gives for $j = 1$ and $k = 3$

$$t_{13} = -\frac{1}{2}\frac{q_1'}{q_1} + O\big(\frac{q_0'}{q_0}\epsilon_2\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O\big(\frac{q_1'}{q_1}\epsilon_3\big) + O\big(\frac{p_0'}{0_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_3\big). \quad (53)$$

Again by (14), (15), (17), (18) and (35)

$$\frac{(\lambda_1+\lambda_2)(q_0'\lambda_1\lambda_3+q_1')}{2(\lambda_1-\lambda_2)m_1} = -\frac{1}{2}\frac{q_0'}{q_0}\epsilon_3\big\{1+O(\epsilon_1)+O(\epsilon_2)\big\} - \frac{1}{2}\frac{q_1'}{q_1}\big\{1+O(\epsilon_1)+O(\epsilon_2)\big\} \quad (54)$$

and

$$\frac{(p_0'\lambda_1\lambda_2+p_1')}{(\lambda_1-\lambda_2)m_1} = \frac{p_0'}{p_0}\epsilon_2\big\{1+O(\epsilon_1)+O(\epsilon_2)\big\} + \frac{p_1'}{p_1}\epsilon_2\big\{1+O(\epsilon_1)+O(\epsilon_2)\big\} \quad (55)$$

Hence by (54) and (55), (33) gives for $j=1$ and $k=2$

$$t_{12} = -\frac{1}{2}\frac{q_1'}{q_1} + O\big(\frac{q_0'}{q_0}\epsilon_3\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O\big(\frac{p_0'}{p_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_2\big). \quad (56)$$

Now, by (14), (15), (17), (18) and (36)

$$\frac{(\lambda_1+\lambda_2)(q_0'\lambda_1\lambda_2+q_1')}{2(\lambda_2-\lambda_1)m_2} = O\big(\frac{q_0'}{q_0}\epsilon_1\epsilon_2\big) - \frac{1}{2}\frac{q_1'}{q_1}\big\{1+O(\epsilon_1)+O(\epsilon_2)\big\}, \quad (57)$$

and

$$\frac{(p_0'\lambda_1\lambda_2+p_1')}{(\lambda_2-\lambda_1)m_2} = O\big(\frac{p_0'}{p_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_2\big). \quad (58)$$

Hence by (57) and (58), (33) gives for $j=2$ and $k=1$

$$t_{21} = -\frac{1}{2}\frac{q_1'}{q_1} + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + O\big(\frac{q_0'}{q_0}\epsilon_1\epsilon_2\big) + O\big(\frac{p_0'}{p_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_2\big). \quad (59)$$

Similar work can be done for the other elemnets t_{jk}, so we obtain

$$t_{23} = \frac{1}{2}\big(\frac{q_0'}{q_0}+\frac{q_1'}{q_1}\big) - \frac{p_0'}{p_0} + O\big(\frac{q_0'}{q_0}\epsilon_1\big) + O\big(\frac{q_0'}{q_0}\epsilon_2\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_2\big) + {} + O\big(\frac{p_0'}{p_0}\epsilon_1\big) + + O\big(\frac{p_0'}{p_0}\epsilon_2\big) + O\big(\frac{p_1'}{p_1}\epsilon_3\big), \quad (60)$$

$$t_{31} = O\big(\frac{q_0'}{q_0}\epsilon_3\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big) + O\big(\frac{p_0'}{p_0}\epsilon_3\big) + O\big(\frac{p_1'}{p_1}\epsilon_1\epsilon_3\big), \quad (61)$$

and

$$t_{32} = O\big(\frac{p_0'}{p_0}\epsilon_1\big) + O\big(\frac{p_1'}{p_1}\epsilon_1\epsilon_3\big) + O\big(\frac{q_0'}{q_0}\epsilon_1\big) + O\big(\frac{q_1'}{q_1}\epsilon_1\big). \tag{62}$$

We note that all the O-terms in t_{jk} $(1 \le j,k \le 3)$ are $L(a,\infty)$ by (34) and (47). Now, by (48), (49), (50), (53) , (56), (60)-(62) we can write the system (30) as

$$Z' = (\Lambda + R + S)Z \tag{63}$$

where

$$R = \begin{pmatrix} -\eta & \eta & \eta \\ \eta & -\eta & -\frac{1}{2}\theta - \eta \\ 0 & 0 & \theta \end{pmatrix}. \tag{64}$$

where

$$\eta = \frac{1}{2}\frac{q_1'}{q_1}, \quad \theta = \frac{q_0'}{q_0} - 2\frac{p_0'}{p_0} \tag{65}$$

and S is $L(a,\infty)$.

4. The Main Result

Now we deal with the critical case which is given by (2). We have the following theorem

Theorem 1. *Let the coefficients q_0, q_1 and p_0 in (1) be $C^{(2)}[a,\infty)$ and let p_1 be $C^{(1)}[a,\infty)$. Let (9), (10) and (34) hold. Let*

$$\frac{q_1'}{q_1} = \sigma_1 \frac{p_1}{q_1}(1+\phi), \tag{66}$$

and

$$\frac{(q_0 p_0^{-2})'}{q_0 p_0^{-2}} = \omega_1 \frac{p_1}{q_1}(1+\psi), \tag{67}$$

where σ_1 and ω_1 are nonzero constants with $\phi(x) \to 0$ and $\psi(x) \to 0$ as $x \to \infty$. Also let

$$\phi'(x) \,\text{and}\, \psi'(x) \,\text{be}\, L(a,\infty). \tag{68}$$

Let

$$\mathrm{Re}(\nu_j - \nu_k) \,\text{have one sign in}\, (a,\infty), \tag{69}$$

for each unequal pair (j,k) where

$$\nu_1 = \lambda_1 - \eta, \nu_2 = \lambda_2 - \eta, \nu_3 = \lambda_3 + \theta \tag{70}$$

Then (1) has solutions

$$y_j(x) \sim q_1^{-1/2}(x)\exp\Big(\int_a^x \lambda_j(t)dt\Big)\ (j=1,2), \tag{71}$$

$$y_3(x) \sim q_0(x)p_0^{-2}(x)\exp\Big(\int_a^x \lambda_3(t)dt\Big). \tag{72}$$

Proof. As in [1], we can apply the asymptotic theorem in Section 2 of [3] to (63) provided only that Λ and R satisfy the conditions in Section 2 of [3]. We shall use (64), (65), (66) and (67). We first require that

$$\frac{q_1'}{q_1} = o\{(\lambda_i-\lambda_j)\},\ \frac{(q_0p_0^{-2})'}{q_0p_0^{-2}} = o\{(\lambda_i-\lambda_j)\},\ i\neq j, 1\leq i,j\leq 3, \tag{73}$$

this being [3] for our system. By (66), (67), (9), (10), (14)-(16), this requirement holds. Second we require that

$$\{\frac{q_1'}{q_1}(\lambda_i-\lambda_j)^{-1}\}'\in L(a,\infty),\ \Big\{\frac{(q_0p_0^{-2})'}{q_0p_0^{-2}}(\lambda_i-\lambda_j)^{-1}\Big\}'\in L(a,\infty)\ \text{for}\ i\neq j, \tag{74}$$

this being [3] for our system. By (66), (67), (14)-(16), this requirement is implied by (47) and (68). Also the eigenvalues μ_j, $(1\leq j\leq 3)$ of $\Lambda+R$ by (1.6.36) from [5] are of the form

$$\mu_j = \lambda_j - \eta + O(\max_{j\neq l}|R|^2|\lambda_l-\lambda_j|^{-1}),\ j=1,2. \tag{75}$$

and

$$\mu_3 = \lambda_3 + \theta + O(\max_{j\neq l}|R|^2|\lambda_l-\lambda_j|^{-1}), \tag{76}$$

where the O-term is $L(a,\infty)$ by (34) and (68). Then by (75) and (76)

$$\mathrm{Re}(\mu_j-\mu_k) = \mathrm{Re}(\nu_j-\nu_k) + O(\max_{j\neq l}|R|^2|\lambda_l-\lambda_j|^{-1}). \tag{77}$$

Hence, by (34), (68) and (69), the dichotomy condition I-II in Levinson's theorem [3, 8] holds. Since (63) has three linearly independent solutions,

$$Z_k(x) = \{e_k + o(1)\}\exp\Big(\int_a^x \nu_k(t)dt\Big),\ k=1,2,3, \tag{78}$$

where e_k is the coordinate vector with k-th component unity and other components zero. We now transform back to Y by means of (29), (21) and (22) and making use of (70), and carrying out the integration of $-\frac{1}{2}(q_1'/q_1)$ and $(q_0p_0^{-2})'/(q_0p_0^{-2})$, we obtain (71) and (72) after an adjustement of a constant multiple in y_k, $1\leq k\leq 3$.

$\square$

5. Examples

5.1. Example 1

We consider Thorem 1 as applied to the coefficients

$$q_0(x) = c_1 x^{\alpha_1}, \quad q_1(x) = c_2 x^{\alpha_2}, \quad p_0(x) = c_3 x^{\alpha_3}, \quad p_1(x) = c_4 x^{\alpha_4},$$

with α_i and c_i $(1 \leq i \leq 4)$, are non-zero real constants. Then it is easy to check that all the conditions (9), (10) and (34) all hold under the two conditions

$$\alpha_1 + \alpha_2 < 2\alpha_3, \ \alpha_3 + \alpha_4 < 2\alpha_2. \tag{79}$$

Also the critical case (66)-(67) is given by

$$\alpha_1 - \alpha_4 = 1, \tag{80}$$

and the nonzero constants σ_1 and ω_1 are given by

$$\sigma_1 = \frac{c_1\alpha_1}{c_4}, \ \omega_1 = \frac{c_1}{c_4}(\alpha_1 - 2\alpha_3), \ (\alpha_1 \neq 2\alpha_3). \tag{81}$$

Also,

$$\phi(x) = \psi(x) = 0 \tag{82}$$

in (66)-(67).

5.2. Example 2

A more general example when the coefficients are

$$q_0 = c_1 x^{\alpha_1} \exp(-x^a), \quad q_1 = c_2 x^{\alpha_2} \exp(4x^a),$$

$$p_0 = c_3 x^{\alpha_3} \exp(2x^a), \quad p_1 = c_4 x^{\alpha_4} \exp(4x^a),$$

where α_i, c_i $(1 \leq i \leq 4)$ and a are real constants with $c_i \neq 0$ and $a > 0$. Then all conditions (9), (10 and (34) are all satisfied. Also the critical case (66)-(67) is given by

$$\alpha_4 - \alpha_2 = a - 1. \tag{83}$$

The values of σ and ω are given by

$$\omega_1 = \frac{-5ac_2}{c_4}, \ \sigma_1 = \frac{4ac_2}{c_4}. \tag{84}$$

Now, in full, (66) and (67) are

$$4ax^{a-1} + \alpha_2 x^{-1} = \sigma_1 \frac{c_4}{c_2} x^{a-1}(1+\phi), \tag{85}$$

and

$$(\alpha_1 - 2\alpha_2)x^{-1} - 5ax^{a-1} = \omega_1 \frac{c_4}{c_2} x^{a-1}(1+\psi), \tag{86}$$

giving

$$\phi(x) = \frac{1}{4}\alpha_2 a^{-1} x^{-a}, \tag{87}$$

and

$$\psi(x) = -\frac{1}{5} c_4 a^{-1}(\alpha_1 - 2\alpha_3)x^{-a}. \tag{88}$$

Then $\phi(x)$ and $\psi(x)$ tend to zero as $x \to \infty$ and $\phi'(x)$ with $\psi'(x)$ are $L(a, \infty)$ and so (68) holds.

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