

ON HYERS-ULAM STABILITY OF NONLINEAR SECOND
ORDER ORDINARY AND FUNCTIONAL
DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we consider the Hyers-Ulam stability of some nonlinear second order ordinary and functional differential equations. As mathematical technique, we use some nonlinear extension of the Grönwall integral inequality.

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1. Introduction

Hyers-Ulam stability which started with Ulam [19] at a wide range talk given before the Mathematics-Club of the University of Wincosin in 1940 on many important unsolved problems had attracted the interest of many researchers. Since then several authors have investigated the stability of linear differential equations, some of these authors include: Hyers [11], Alsina and Ger [1], Miura

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et al [8,9], Takahasi et al [17,18].

However, only few authors have considered the Hyers-Ulam stability of nonlinear differential equations. These include: Rus[15,16], Ravi et al[14], Jinghao[6], Qarawani[11,12], Qusuay et al[10] and Motaza and Omid[7]. Motivation for this study came from the work of Qusuay et al[10] where Hyers-Ulam stability of nonlinear differential equations of second order was considered using Grönwall lemma.

In this paper, we consider the Hyers-Ulam stability of the nonlinear differential equation

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad (1.1)$$

where $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$, $\mathbf{I} = [t_0, b)(b < \infty)$ and $f \in C(\mathbf{I} \times \mathbf{R}_+^2, \mathbf{R}_+)$

2. Preliminaries

In this section, we give some definitions, theorems, and lemma which will be useful in subsequent discussion.

Lemma 2.1. (see Bihari [2,3]) *Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality*

$$u(t) \leq K + M \int_a^t f(s)\omega(u(s))ds, \quad a \leq t < b \quad (2.1)$$

implies the inequality

$$u(t) \leq \Omega^{-1} \left(\Omega(k) + M \int_a^t f(s)ds \right), \quad a \leq t \leq b' \leq b \quad (2.2)$$

where

$$\Omega(u) = \int_{u_0}^u \frac{dt}{\omega(t)}, \quad 0 < u_0 < u \quad (2.3)$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω for t in the subinterval $[a, b']$ of $[a, b]$ such that

$$\Omega(k) + M \int_a^t f(s)ds \in \text{Dom}(\Omega^{-1}).$$

Definition 2.2. A function ω is said to belong to a class S if it satisfies the following conditions:

- i $\omega(u) > 0$ is nondecreasing and $\omega \in C(\mathbf{R}_+, \mathbf{R}_+)$ for $u > 0$
- ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$
- iii there exists a function ϕ , continuous on $[0, \infty)$ with $\omega(u) \leq \omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha > 0, u > 0$.

Theorem 2.3. (see [4]) Suppose that

- i $u(t), r(t), g(t) \in C(\mathbf{I}, \mathbf{R}_+)$
- ii $\omega(u)$ is a nonnegative, monotonic nondecreasing, continuous, submultiplicative function for $u > 0$

if

$$u(t) \leq K + \int_{t_0}^t r(s)u(s)ds + \int_{t_0}^t g(s)\omega(s)ds, \quad t \in \mathbf{I} \tag{2.4}$$

for $K > 0$, a constant, then

$$u(t) \exp\left(-\int_{t_0}^t r(s)ds\right) \leq \Omega^{-1}\left(\Omega(K) + \int_{t_0}^t g(s)\omega\left(\exp\int_{t_0}^s r(\delta)d\delta\right) ds\right) \quad t \in \mathbf{I} \tag{2.5}$$

where

$$\Omega(u) = \int_{u_0}^u \frac{ds}{\omega(s)}, \quad 0 < u_0 \leq u \tag{2.6}$$

and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0, b) \in \mathbf{I}$ so that

$$\Omega(K) + \int_{t_0}^t g(s)\omega\left(\exp\int_{t_0}^s r(\delta)d\delta\right) ds \in Dom(\Omega^{-1}). \tag{2.7}$$

Definition 2.4. Equation (1.1) is said to be Hyers-Ulam stable if there exists constants $K > 0, \epsilon > 0$ and the solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of

$$|u''(t) + f(t, u(t), u'(t))| \leq \epsilon \tag{2.8}$$

satisfies

$$|u(t) - u_0(t)| \leq K\epsilon.$$

where $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is a solution of equation (1.1) and K is the Hyers-Ulam constant.

3. Main Result

Our main results are presented in the following theorems.

In the first theorem we consider the Hyes-Ulam stability of a second order differential equation which is nonlinear in only $u(t)$

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) = 0 \quad (3.1)$$

where $a, b, g, \in C(\mathbf{I}, \mathbf{R}_+)$. and $f \in C(\mathbf{R}_+, \mathbf{R}_+)$ with $f \in S$.

Theorem 3.1. *Suppose that $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the differential inequality:*

$$|u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t))| \leq \epsilon \quad (3.2).$$

Then, if

i $\int_{t_0}^t \frac{1}{b(s)} ds \leq p$ for $p > 0$, and all $t \in \mathbf{R}_+$

ii $\int_{t_0}^t \left(\frac{a(s)}{b(s)} - 1 \right) |u'(s)| ds \leq m$. for $m \geq 0$

iii $|u'(t)| \leq \lambda$ for $\lambda \geq 0$.

iv $\lim_{t \rightarrow \infty} \int_{t_0}^t \frac{1}{b^2(s)} ds = M < \infty$ for $M > 0$

$\lim_{t \rightarrow \infty} \int_{t_0}^t \frac{g(s)}{b(s)} ds = T < \infty$ for $T > 0$

are satisfied, equation (3.1) is Hyers-Ulam stable with the Hyers-Ulam constant K defined as

$$K = (|E(t_0)| + \lambda + p + m) M \Omega^{-1} (\Omega(1) + T \omega(\exp M)) \quad (3.3)$$

Proof. Inequality (3.2), implies that

$$-\epsilon \leq u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) \leq \epsilon \quad (3.4)$$

it follows that

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) \leq \epsilon \quad (3.5)$$

Define

$$E(t) = \frac{u'(t)}{b(t)} + u(t), \quad u(t) \neq 0, \quad b(0) \neq 0 \quad (3.6)$$

clearly

$$E(t) = E(t_0) + \int_{t_0}^t \frac{d}{ds} \left(\frac{u'(s)}{b(s)} + u(s) \right) ds \quad (3.7)$$

Where

$$E(t_0) = \frac{u'(t_0)}{b(t_0)} + u(t_0)$$

$$E(t) = E(t_0) + \int_{t_0}^t \left(u'(s) + \frac{u''(s)}{b(s)} - \frac{db(s)}{ds} \frac{u'(s)}{b^2(s)} \right) ds \quad (3.8)$$

Since $b(t)$ is an increasing function, $\frac{db(t)}{dt} \geq 0$.

It follows from(3.8) that

$$E(t) \leq E(t_0) + \int_{t_0}^t \left(u'(s) + \frac{u''(s)}{b(s)} \right) ds \quad (3.9)$$

Substituting for $u''(t)$ in (3.9) using (3.5), we have

$$E(t) \leq E(t_0) + \int_{t_0}^t \left(u'(s) - \frac{1}{b(s)} \left(a(s)u'(s) + \frac{1}{b(s)}u(s) + g(s)f(u(s)) - \epsilon \right) \right) ds$$

$$E(t) \leq E(t_0) - \int_{t_0}^t (-u'(s) + \frac{a(s)}{b(s)}u'(s) + \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)}) ds$$

$$E(t) \leq E(t_0) - \int_{t_0}^t \left(\left(\frac{a(s)}{b(s)} - 1 \right) u'(s) + \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)} \right) ds \quad (3.10)$$

Replacing $E(t)$ in (3.10) with (3.6), we have

$$u(t) \leq E(t_0) - \frac{u'(t)}{b(t)} - \int_{t_0}^t \left(\left(\frac{a(s)}{b(s)} - 1 \right) u'(s) \right)$$

$$+ \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)} \Big) ds.$$

Taking the absolute value of both sides, we get

$$\begin{aligned} |u(t)| &\leq |E(t_0)| + \frac{|u'(t)|}{b(t)} \\ &\quad + \int_{t_0}^t \left(\left(\frac{a(s)}{b(s)} - 1 \right) |u'(s)| + \frac{1}{b^2(s)}|u(s)| + \frac{g(s)}{b(s)}f(|u(s)|) + \frac{\epsilon}{b(s)} \right) ds \end{aligned}$$

Using conditions (i-iii) with $\frac{1}{b(t)} \leq 1$, we get

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^t \frac{1}{b^2(s)}|u(s)|ds + \int_{t_0}^t \frac{g(s)}{b(s)}f(|u(s)|)ds \quad (3.11)$$

Setting

$$\frac{1}{b^2(t)} = \alpha(t), \quad \frac{g(t)}{b(t)} = \gamma(t)$$

and using them in (3.11), we have

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^t \alpha(s)|u(s)|ds + \int_{t_0}^t \gamma(s)f(|u(s)|)ds \quad (3.12)$$

with

$$f(|u(t)|) = \omega(|u(t)|) \text{ and } \epsilon \geq 1$$

(3.12) becomes

$$|u(t)| \leq C + \int_{t_0}^t \alpha(s)|u(s)|ds + \int_{t_0}^t \gamma(s)\omega(|u(s)|)ds \quad (3.13)$$

where

$$C = \epsilon (|E(t_0)| + \lambda + p + m)$$

thus, we have

$$\frac{|u(t)|}{C} \leq 1 + \int_{t_0}^t \alpha(s)\frac{|u(s)|}{C}ds + \int_{t_0}^t \gamma(s)\omega\left(\frac{|u(s)|}{C}\right)ds \quad (3.14)$$

the application of theorem 2.3 then yields

$$|u(t)| \leq C\Omega^{-1}(\Omega(1))$$

$$+ \int_{t_0}^t \gamma(s)\omega \left(\exp \int_{t_0}^s \alpha(\delta)d\delta \right) ds \left(\exp \int_{t_0}^t \alpha(s)ds \right)$$

using the condition (iv), we obtain

$$|u(t)| \leq CM\Omega^{-1} (\Omega(1) + T\omega(\exp M)) \tag{3.15}$$

Substituting for C in (3.15) gives

$$|u(t) - u_0(t)| \leq |u(t)| \leq \epsilon (|E(t_0)| + \lambda + p + m) M\Omega^{-1} (\Omega(1) + T\omega(\exp M))$$

Therefore, equation (3.1) is Hyers-Ulam stable with the Hyers-Ulam constant

$$K = (|E(t_0)| + \lambda + p + m) M\Omega^{-1} (\Omega(1) + T\omega(\exp M)).$$

Example. To investigate Hyers-Ulam stability of the second order nonlinear differential equation

$$u''(t) + t^2u'(t) + t^{-2}u(t) + t^{-6}u^2(t) = 0$$

all the conditions (i-iv) of Theorem 3.1 are satisfied so the equations is Hyers-Ulam stable.

Next we consider the Hyers-Ulam stability of a second order differential equation which is nonlinear in both $u(t)$. and $u'(t)$

$$u''(t) + \phi(t)g(u(t))h(u'(t)) = 0 \tag{3.16}$$

together with initial condition

$$u(t_0) = u'(t_0) = 0$$

where $h, \phi \in C(\mathbf{I}, \mathbf{R}_+)$, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$ and $h(u') > 0$.

Theorem 3.2. Let $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfy the differential inequality

$$|u''(t) + \phi(t)g(u(t))h(u'(t))| \leq \epsilon \tag{3.17}$$

for all $t \in \mathbf{I}$ and for some $\epsilon > 0$, then there exists a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of equation(3.17) such that $|u(t) - u_0(t)| \leq K\epsilon$, for

$$K = \frac{1}{\delta\lambda} P\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right) \tag{3.18}$$

with $G \in S$, provided the following conditions are satisfied:

$$i \quad G(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds < \infty$$

$$ii \quad R(u(t)) = \int_{u(t_0)}^{u(t)} \frac{s}{h(s)} ds < \infty$$

$$iii \quad \lim_{t \rightarrow \infty} \int_{t_0}^t |\phi'(s)| ds = M < \infty$$

$$iv \quad \alpha(t) \geq \delta, \quad \text{where } \delta > 0$$

$$v \quad \int_{t_0}^t \frac{|u'(s)|}{|h(u'(s))|} ds \leq P$$

Proof. From (3.17), it follows that

$$-\epsilon \leq u''(t) + \phi(t)g(u(t))h(u'(t)) \leq \epsilon \quad \text{for all } t \geq t_0 \quad (3.19)$$

$$u''(t) + \phi(t)g(u(t))h(u'(t)) \leq \epsilon \quad \text{for all } t \geq t_0$$

$$\frac{u''(t)u'(t)}{h(u'(t))} + \phi(t)g(u(t))u'(t) \leq \frac{u'(t)\epsilon}{h(u'(t))} \quad (3.20)$$

Using (i) and (ii), we get

$$\frac{dR(u'(t))}{dt} + \phi(t) \frac{dG(u(t))}{dt} \leq \frac{u'(t)\epsilon}{h(u'(t))} \quad \text{for all } t \geq t_0 \quad (3.21)$$

Integrating by part from t_0 to t

$$R(u'(t)) - \phi(t)G(u(t)) + \int_{t_0}^t \phi'(s)G(u(s))ds \leq \epsilon \int_{t_0}^t \frac{u'(s)}{h(u'(s))} ds$$

Taking the absolute value of both sides, we have

$$|R(u'(t)) - \phi(t)G(u(t))| \leq \epsilon \int_{t_0}^t \frac{|u'(s)|}{h(|u'(s)|)} ds + \int_{t_0}^t \phi'(s)G(|u(s)|)ds \quad (3.22)$$

Setting

$$|R(u'(t)) - \phi(t)G(u(t))| \geq \alpha(t)|u(t)||u'(t)| \quad (3.23)$$

for $\alpha(t) \in C(\mathbf{I}, \mathbf{R}_+)$ So by (3.8), equation (3.7) becomes

$$\alpha(t)|u(t)||u'(t)| \leq \epsilon \int_{t_0}^t \frac{|u'(s)|}{h(|u'(s)|)} ds + \int_{t_0}^t |\phi'(s)|G(|u(s)|)ds \quad (3.24)$$

Using condition (iv), we get

$$\delta|u(t)||u'(t)| \leq \epsilon \int_{t_0}^t \frac{|u'(t)|}{h(|u'(s)|)} ds + \int_{t_0}^t |\phi'(s)|G(|u(s)|)ds \tag{3.25}$$

It follows that

$$|u(t)| \leq \frac{\epsilon}{\delta|u'(t)|} \int_{t_0}^t \frac{|u'(t)|}{|h(u'(s))|} ds + \frac{1}{\delta|u'(t)|} \int_{t_0}^t |\phi'(s)|G(|u(s)|)ds \tag{3.26}$$

Using condition(v) and setting $|u'(t)| \leq \lambda$, for $\lambda \geq 0$, we have

$$|(u(t))| \leq \frac{\epsilon}{\delta\lambda}P + \frac{1}{\delta\lambda} \int_{t_0}^t |\phi'(s)|G(|u(s)|)ds$$

From the fact that $G \in S$ for

$$G(|u(t)|) = \omega(|u(t)|)$$

it follows that

$$|(u(t))| \leq \frac{\epsilon}{\delta\lambda}P + \frac{1}{\delta\lambda} \int_{t_0}^t |\phi'(s)|\omega(|u(s)|)ds \tag{3.27}$$

Setting

$$L = \frac{\epsilon}{\delta\lambda}P \tag{3.28}$$

We have

$$\frac{|(u(t))|}{L} \leq 1 + \frac{1}{\delta\lambda} \int_{t_0}^t |\phi'(s)|\omega\left(\frac{|u(s)|}{L}\right)ds \tag{3.29}$$

using lemma 2.2 and equation(2.3) we obtain

$$\frac{|u(t)|}{L} \leq \Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} \int_{t_0}^t |\phi'(s)|ds \right)$$

By condition (iii), we have

$$\begin{aligned} \frac{|u(t)|}{L} &\leq \Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda}M \right) \\ |u(t)| &\leq L\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda}M \right) \end{aligned} \tag{3.30}$$

Substituting for L from (3.28), equation (3.30) becomes

$$|u(t)| \leq \frac{\epsilon}{\delta\lambda}P\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda}M \right)$$

Since

$$|u(t) - u(t_0)| \leq |u(t)|$$

we have

$$|u(t) - u_0(t)| \leq \frac{\epsilon}{\delta\lambda} P\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right)$$

Hence, equation(3.16) is Hyers-Ulam stable with Hyers-Ulam constant K given as

$$K = \frac{1}{\delta\lambda} P\Omega^{-1} \left(\Omega(1) + \frac{1}{\delta\lambda} M \right).$$

Example. To investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form

$$u''(t) + t^{-3}u^2(t)u'^4(t) = 0,$$

where $-3 \int_{t_0}^t \frac{ds}{s^4} < \infty$ and $h(u'(t)) = u'^4(t)$, $u(t_0) = u'(t_0) = 0$.

References

- [1] C. Alsina, R.N. Ger, On some inequalities and stability result related to the exponential function, *J. Inequal. Appl.*, **2** (1988), 373-38.
- [2] I. Bihari, Researches of the boundedness and stability of the solutions of nonlinear differential equations, *Acta. Math. Acad. Sc. Hung.*, **7** (1957), 278-291.
- [3] I. Bihari, A generalisation of a lemma of Bellman and its application to uniqueness problem of differential equations, *Acta Maths. Acad. Sc. Hung.*, **7** (1956), 71-94.
- [4] U.D. Dhongade, S.G. Deo, Some generalisation of Bellman-Bihari integral inequalities, *Journal of Mathematical Analysis and Applications*, **44** (1973), 218-226.
- [5] D.H. Hyers, On the stability of the linear functional equation, In: *Proceedings of the National Academy of Science of the United States of America*, **27** (1941), 222-224.
- [6] H. Jinghao, S.-M. Jung, L. Yongjin, On Hyers-Ulam stability of nonlinear differential equations, *Bull. Korean Maths. Soc.*, **52** (2015), 685-697.

- [7] G. Mortaza, B. Omid, Hyers-Ulam stability of nonlinear integral equation, *Fixed Point Theory and Applications* (2010), 1-6.
- [8] T. Miura, S. Miyajima, S.E. Yakahasi, A characterisation of Hyers-Ulam stability of first order linear differential operator, *Journal of Mathematical Analysis and Applications*, **286** (2003), 136-1463.
- [9] T. Miura, S. Miyajima, S. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, *Mathematisch Nachrichten*, **258** (2003), 90-96.
- [10] H. Algfary Qusausy, S.M. Jung, On the Hyers-Ulam stability of differential equations of second order, *Abstract and Applied Analysis* (2014), 1-8.
- [11] N.M. Qarawani, On Hyers-ULam stability for nonlinear differential equations of n -th order, *International Journal of Analysis and Application*, **1** (2013), 71-78.
- [12] M.N. Qarawani, On Hyers-Ulam stability of a generalised second order nonlinear differential equations, *Applied Mathematics*, **3** (2012), 1857-1861.
- [13] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society*, **72** (1978), 297-300.
- [14] K. Ravi, R. Murali, A. Ponmanaselvan, R. Veerasivsji, Hyers-Ulam stability of n -th order nonlinear differential equations with intial conditions, *International Journal of Mathematics And its Application*, **4** (2016), 121-132.
- [15] I.A. Rus, Ulam stability of ordinary differential equation, *Studia Universitatis Babeş-Bolyai Mathematical*, **54** (2010), 306-309.
- [16] I.A. Rus, Ulam stability of ordinary differential equations in a Banach space, *Carpathian, J. Math.*, **26** (2010), 103-107.
- [17] S.-M. Takahasi, H. Miura Takagi, S. Miyajima, The Hyers-Ulam stability constants of first order linear differential operators, *Journal of Mathematical Analysis and Applications*, **296** (2004), 403-409.
- [18] S. Miura Takahasi, S. Miyajima, On the Hyers-UIam stability of the Banach space-valued differential equation $y' = \lambda y$, *Bulletin of the Korean Mathematical Society*, **392** (2002), 309-315.

- [19] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wily, New York, USA, 1960.