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CLASSIFICATION OF SOLUTIONS OF SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: In this paper we classify all the solutions of a second order neutral differential equation with "maxima" into four classes and obtain conditions for the existence/nonexistence of solutions in these classes.

Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we study the oscillatory and asymptotic behavior of solutions of the second order nonlinear neutral delay differential equation with "maxima" of the form

$$(a(t)(x(t)+p(t)x(\tau(t)))')'+q(t)\max_{[\sigma(t),\mu(t)]}x^{\alpha}(s)=0,\quad t\geq t_0, \tag{1.1}$$

where $a(t)$ is positive and differentiable. $p(t)$ and $q(t), \tau(t), \sigma(t)$ and $\mu(t)$ are continuous functions and $\sigma(t) \leq \mu(t) \leq t$, and α is a ratio of odd positive integers.

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By a proper solution of equation (1.1) we mean a function $x(t) \in C([t_0, \infty), R)$ which satisfies equation (1.1) for all $t \geq t_0$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_0$. We assume that equation (1.1) possesses proper solutions. A proper solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. A nonoscillatory solution x of equation (1.1) is said to be weakly oscillatory if x' is oscillatory.

The study of differential equations with "maxima" begins with the words of A.Magomedov [6], [3] where a linear differential equation with "maxima" was considered as a mathematical model in the theory of automatic control. In most cases one uses "maxima" in the right hand side when the control corresponds to the maximal deviation of the regulated quantity that could be, for example, pressure,density,temperature,heat etc.

The oscillatory behavior of solutions of differential equations with "maxima" where studied by Bainov and his associates, see for example [1,2,5,9 and 10] and the references cited therein. However there are few papers dealing the oscillatory behavior of solutions of second order neutral delay differential equations with "maxima", see [4,9,10].

The aim of this paper is to consider the cases $q \geq 0$ and q changes the sign for all large t, to give sufficient conditions in order that every solution of equation (1.1) is either oscillatory and/or weakly oscillatory,and to study the asymptotic nature of nonoscillatory solutions of equation (1.1). With respect to their asymptotic behavior,all the solutions of equation (1.1) may be priori divided into the following classes:

$$\begin{aligned} M^{+} =&\{x = x(t) \text{ is a soluion of (1.1): there exists } t_x \geq t_0; x(t)x'(t) \geq 0 \\ &\text{for all } t \geq t_x\}, \\ M^{-} =&\{x = x(t)\text{is a soluion of (1.1): there exists } t_x \geq t_0; x(t)x'(t) \leq 0 \\ &\text{for all } t \geq t_x\}, \\ OS =&\{x = x(t) \text{ is a solution of (1.1): there exist } \{t_n\},\ t_n \to \infty; x(t_n) = 0\}, \\ WOS =&\{x = x(t) \text{ is a nonoscillatory solution of (1.1): } x'(t) \\ &\text{is oscillatory } t \geq t_x\}. \end{aligned}$$

With very simple argument one can prove that M^+, M^-, OS and WOS are mutually disjoint. In Section 2, we establish sufficient conditions for the existence / nonexistence of solutions in these classes, and the asymptotic behavior of solutions in the class M^+, M^- are investigated in Section 3. Examples are provided to illustrate the main results.

2. Existence and Oscillation of Solutions

First we examine the existence of solutions of equation (1.1) in the class M^+.

Theorem 2.1. *With respect to the differential equation* (1.1) *assume that*

$$p(t) \geq 0 \text{ and non decreasing for all } t \geq t_0, \tag{2.1}$$

$$\mu(t) \text{ and } \tau(t) \text{ are nondecreasing for all } t \geq t_0, \tag{2.2}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty \tag{2.3}$$

hold. Then for equation (1.1), *we have* $M^+ = \phi$.

Proof. Assume that equation (1.1) has a solution $x \in M^+$. Without loss of generality we may assume that $x(t) > 0$, $x'(t) \geq 0$, $x(\sigma(t)) > 0$, $x'(\sigma(t)) \geq 0$, $x(\mu(t)) > 0$, $x'(\mu(t)) \geq 0$ for all $t \geq t_1 \geq t_0$ (the proof is similar if $x(t) < 0$, $x(\sigma(t)) < 0$, $x(\mu(t)) < 0$ for all large t).

Let

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.4}$$

Then $z(t) > 0$ for all $t \geq t_1$ and

$$z'(t) = x'(t) + p'(t)x(\tau(t)) + p(t)x'(\tau(t))\tau'(t).$$

By conditions (2.1) and (2.2), we have $z'(t) > 0$ *for all* $t > t_1$. Since $x \in M^+$, we have $\max_{[\sigma(t),\mu(t)]} x^\alpha(s) = x^\alpha(\mu(t))$ and the equation (1.1) becomes

$$(a(t)z'(t))' + q(t)x^\alpha(\mu(t)) = 0$$

or

$$(a(t)z'(t))' = -q(t)x^\alpha(\mu(t)) \leq 0 \text{ for } t \geq t_1. \tag{2.5}$$

Now

$$\left(\frac{a(t)z'(t)}{x^\alpha(\mu(t))}\right)' = \frac{(a(t)z'(t))'}{x^\alpha(\mu(t))} - \frac{\alpha a(t)z'(t)}{x^{\alpha+1}(\mu(t))}x'(\mu(t))\mu'(t). \tag{2.6}$$

Since $z'(t) > 0, x'(\mu(t)) \geq 0$ *and* $\tau'(t) \geq 0$, we have from (2.5) and (2.6)

$$\left(\frac{a(t)z'(t)}{x^\alpha(\mu(t))}\right)' \leq -q(t).$$

Integrating the last inequality from t_1 *to* t, we obtain

$$\left(\frac{a(t)z'(t)}{x^{\alpha}(\mu(t))}\right) - \left(\frac{a(t_1)z'(t_1)}{x^{\alpha}(\mu(t_1))}\right) \leq - \int_{t_1}^{t} q(s)ds. \tag{2.7}$$

Letting $t \to \infty$ and taking lim sup, we obtain $z'(t) < 0$ by (2.3) for all $t \geq t_1$, which is a contradiction. This completes the proof. □

Remark 2.1 Theorem 2.1 does not require that α be greater than one or less than one.

The following example shows that assumption (2.3) cannot be dropped without violating the validity of Theorem 2.1.

Example 2.1. Consider the neutral differential equation

$$[t(x(t) + \frac{t-1}{t}x(t-1))']' + \frac{3}{t(t-1)} \max_{[t-2,t]} x(s) = 0, \ t \geq 2. \tag{2.8}$$

All conditions of Theorem 2.1 are satisfied,except condition (2.3). In fact,equation (2.8) has the solution $x(t) = \frac{t-1}{t} \in M^+$.

In the next theorem we consider the existence of solutions in the class M^+ when

$$-1 \leq p_1 \leq p(t) \leq 0; \tag{2.9}$$

$$q(t) \geq 0 \ for \ all \ t \geq t_0; \tag{2.10}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty; \tag{2.11}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{a(s)}ds = \infty. \tag{2.12}$$

Theorem 2.2. *With respect to the differential equation* (1.1), *assume* (2.2) *and* (2.9) *to* (2.12) *hold. Then for equation* (1.1) *we have* $M^+ = \phi$.

Proof. Proceeding as in the proof of Theorem 2.1, we have (2.4) and (2.5). Since $x \in M^+$, we obtain

$$z(t) \geq (1 + p(t))x(\tau(t)) > 0.$$

From (2.5), we have $a(t)z'(t)$ is of one sign for all $t \geq t_1 \geq t_0$. If $a(t)z'(t) < 0 \ for \ t \geq t_1, then \ a \ (t)z'(t) \leq a(t_1)z'(t_1) < 0 \ for \ t \geq t_1$. Dividing the last inequality by $a(t)$ and then integrating the resulting inequality from t_1 *to* t we obtain

$$z(t) \leq z(t_1) + a(t_1)z'(t_1) \int_{t_1}^{t} \frac{1}{a(s)} ds.$$

Letting $t \to \infty$ in the above inequality we obtain $z(t) \to -\infty$, which is a contradiction.Hence $z'(t) > 0$ for $t \geq t_1$. Now proceeding as in the proof of Theorem 2.1, we obtain,by using (2.11) that $z'(t) < 0$ for $t \geq t_1$, which is a contradiction. This completes the proof. □

The following example shows that some assumptions of Theorem 2.2 cannot be dropped without violating the validity of Theorem 2.2.

Example 2.2 Consider the neutral delay differential equation of the form

$$[t^2(x(t) - 2x(t-1))']' + 2 \max_{[t-2,t]} x(s) = 0, \ t \geq 1. \tag{2.13}$$

All conditions of Theroem 2.2 are satisfied except the conditions (2.9) and (2.12).In fact, equation (2.13) has a solution $x(t) = t \ \in M^+$.

Next we examine the problem of existence of solutions of equation in the class M^-.

Theorem 2.3. *With respect to the differential equation* (1.1) *assume that* $\sigma(t)$ *is nondecreasing and* $\sigma(t) \geq \tau(t)$. *If*

$$0 < \alpha < 1 \tag{2.14}$$

$$p(t) \geq 0 \text{ and nondecreasing for all } t \geq t_0, \tag{2.15}$$

and

$$\limsup_{t\to\infty} \int_T^t \max_{[\sigma(t),\mu(t)]} \frac{1}{1+p(s)a(s)} \left(\int_T^s q(u)du \right) ds = \infty, \tag{2.16}$$

then for equation (1.1) *we have* $M^- = \phi$.

Proof. Assume that equation (1.1) has a solution $x \in M^-$. Without loss of generality we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$,$x'(t) \leq 0, x(\sigma(t)) > 0, x'(\sigma(t)) \leq 0, x(\tau(t)) > 0, x'(\tau(t)) \leq 0$, for all $t \geq t_1$, since the proof is similar if $x(t) < 0$ *and* $x'(t) \geq 0$ for all large t. Let $z(t) = x(t) + p(t)x(\tau(t))$,then in view of the hypotheses $z(t) > 0$ and $z'(t) \leq 0$ $for\ all\ t \geq t_1$.

Since $x \in M^-$ we have $\max_{[\sigma(t),\mu(t)]} x^\alpha(s) = x^\alpha(\sigma(t))$ and the equation (1.1) becomes

$$(a(t)z'(t))' + q(t)x^\alpha(\sigma(t)) = 0. \tag{2.17}$$

Now

$$\left(\frac{a(t)z'(t)}{x^\alpha(\sigma(t))}\right)' = \frac{(a(t)z'(t))'}{x^\alpha(\sigma(t))} - \alpha\frac{a(t)z'(t)}{x^{\alpha+1}(\sigma(t))}x'(\sigma(t))\sigma'(t).$$

Since $z'(t) \leq 0,\ x'(\sigma(t)) \leq 0$ *and* $\sigma'(t) \geq 0$, we have

$$\left(\frac{a(t)z'(t)}{x^\alpha(\sigma(t))}\right)' \leq -q(t).$$

Integrating the last inequality from t_1 *to* t, we obtain

$$\frac{a(t)z'(t)}{x^\alpha(\sigma(t))} - \frac{a(t_1)z'(t_1)}{x^\alpha(\sigma(t_1))} \leq -\int_{t_1}^{t} q(s)ds$$

or

$$\frac{z'(t)}{x^\alpha(\sigma(t))} \leq -\frac{1}{a(t)}\int_{t_1}^{t} q(s)ds. \tag{2.18}$$

Since $x(t) \in M^-$,we have

$$z(t) = x(t) + p(t)x(\tau(t)) \leq (1+p(t))x(\tau(t)) \leq (1+p(t))x(\sigma(t)). \tag{2.19}$$

From (2.18) and (2.19), we obtain

$$\frac{z'(t)}{z^\alpha(t)} \leq -\frac{1}{a(t)(1+p(t))^\alpha}\int_{t_1}^{\sigma} q(s)ds.$$

Integrating the last inequality from t_1 *to* t, we obtain

$$\int_{t_1}^{t}\frac{1}{a(s)(1+p(s))^\alpha}\int_{t_1}^{s} q(u)duds \leq z^{-\alpha+1}(t) - z^{-\alpha+1}(t_1) \leq z^{-\alpha+1}(t) < \infty$$

since $0 < \alpha < 1$. Letting $t \to \infty$,we obtain a contradiction with (2.16). The proof is now complete. □

The following example shows that the assumption (2.16) cannot be dropped without violating the validity of Theorem 2.3.

Example 2.3. Consider the neutral differential equation

$$[e^{2t}(x(t)+\frac{1}{e}x(t-1))']'+2e^{\frac{4t-2}{3}}\max_{[t-2,t]}x^{\frac{1}{3}}(s)=0,\ t\geq 1. \tag{2.20}$$

All conditions of Theroem 2.3 are satisfied except the condition (2.16). In fact $x(t)=e^{-t}\in M^-$ is one such solution of equation (2.20).

Next we obtain a conditions for the solutions of equation (1.1) to be weakly oscillatory or oscillatory.

Theorem 2.4. *With respect to the differential equation* (1.1) *assume that* (2.11), (2.12) *and* (2.16) *hold. If*

$$\sigma(t) \text{ and } \tau(t) \text{ are nondecreasing for all } t\geq t_1. \tag{2.21}$$

$$p(t)\equiv p \tag{2.22}$$

then $WOS\cup OS\neq\phi$.

Proof. From Theorem 2.1 it follows that for equation (1.1) the class $M^+=\phi$. To complete the proof it is enough to show that $M^-=\phi$. for the equation (1.1). Assume that $x\in M^-$. Without loss of generality we may assume that there exists $t_1\geq t_0$ such that $x(t)>0,\ x'(t)\leq t_0, x(\sigma(t))>0, x'(\sigma(t))\leq 0,\ for\ all\ t\geq t_1$. (The proof is similar that $x(t)<0$,$x'(t)\geq t_0\ for\ all\ large\ t$.) Let $z(t)=x(t)+px(\tau(t)).Then\ inview\ of(2.21)and(2.22)\ z(t)>0\ and\ z'(t)\leq 0\ for\ all\ t\geq t_1$. Proceeding as in Theorem 2.3 we obtain

$$w(t)=w(t_1)+\int_{t_1}^{t}w(s)\left(\frac{-\alpha x'(\sigma(s))\sigma'(s)}{x(\sigma(s))}\right)ds-\int_{t_1}^{t}q(s)ds$$

$$\leq w(t_1)+\int_{t_1}^{t}w(s)\left(\frac{-\alpha x^{\alpha-1}(\sigma(s))x'(\sigma(s))(\sigma'(s)}{x^{\alpha}(\sigma(s))}\right)ds,$$

where $w(t)=\frac{a(t)z'(t)}{x^{\alpha}(\sigma(t))}$ *and* $w(t_1)<0$. By using Gromwall's inequality, we obtain

$$w(t)\leq\frac{w(t_1)x^{\alpha}(\sigma(t_1))}{x^{\alpha}(\sigma(t))}$$

or

$$a(t)z'(t)\leq w(t)x^{\alpha}(\sigma(t_1))=K\ (K<0).$$

Thus,for all large t, we have

$$z(t) - z(t_1) < K \int_{t_1}^{t} \frac{1}{a(s)} ds$$

which,because of (2.12) implies that $z(t) \to -\infty \ as \ t \to \infty$, a contradiction.This complete the proof. □

3. Behavior of Solutions in M^+ and M^-

In this section, we study the asymptotic behavior of the eventually monotone solutions of equation (1.1).

Theorem 3.1. *With respect to the differential equation* (1.1) *asume that* (2.15) *and* (2.16) *hold. Then for every solution* $x \in M^+$ *we have* $\lim_{t\to\infty} x(t) = 0$.

Proof. From the proof of Theorem 2.3, we have

$$\int_{t_1}^{t} \frac{1}{a(s)(1+p(s))^{\alpha}} \left(\int_{t_1}^{s} q(u)du \right) ds \le \frac{1}{z^{\alpha-1}(t)} - \frac{1}{z^{\alpha-1}(t_1)}$$

By condition (2.16), we have $z(t) \to 0 \ as \ t \to \infty$. This completes the proof. □

Finally, we examine the asymptotic behavior of solutions in the class M^+.

Theorem 3.2. *With respect to the differential equation* (1.1) *assume that conditions* (2.1) *and* (2.2) *hold with* $p(t)$ *is bounded.If*

$$\lim_{t\to\infty} \sup \int_{t_1}^{t} q(s) \left(\int_{t_1}^{s} \frac{1}{a(u)} du \right) ds = \infty, \ t_1 \ge t_0. \tag{3.1}$$

then every solution in the M^+ *is unbounded.*

Proof. Let $x(t)$ be a solution of equation (1.1) such that $x(t) \in M^+$. Then proceeding as in the proof of Theorem 2.1, we have $z(t) > 0, z'(t) > 0$ for all $t \ge t_1$. Define

$$w(t) = \frac{-a(t)z'(t)}{x^{\alpha}(\mu(t))} \int_{t_1}^{t} \frac{1}{a(s)} ds$$

then

$$w'(t) = q(t)\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{z'(t)}{x^{\alpha}(\mu(t))} + \frac{\alpha a(t)z'(t)x'(\mu(t))\mu'(t)}{x^{\alpha+1}(\mu(t))}\int_{t_1}^{t}\frac{1}{a(s)}ds$$

$$\geq q(t)\int_{t_1}^{t}\frac{1}{a(s)}ds - \frac{z'(t)}{x^{\alpha}(\mu(t)}.$$

Integrating the last inequality from t_1 *to* t, we obtain

$$w(t) \geq \int_{t_1}^{t} q(s)\int_{t_1}^{s}\frac{1}{a(u)}duds - \int_{t_1}^{t}\frac{z'(s)}{x^{\alpha}(\mu(s))}ds. \tag{3.2}$$

Since the function $\frac{z'(t)}{x^{\alpha}(\mu(t))}$ is positive for $t \geq t_1$ then $\lim\limits_{t\to\infty}\int_{t_1}^{t}\frac{z'(s)}{x^{\alpha}(\mu(s))}ds$ exists. Assume that

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z'(s)}{x^{\alpha}(\mu(s))}ds = K < \infty.$$

Taking into account (3.1), and (3.2) we obtain $\lim\limits_{t\to\infty}\sup w(t) = \infty$ which gives a contradiction,since $w(t)$ is negative for all $t \geq t_1$. Thus

$$\int_{t_1}^{t}\frac{z'(s)}{x^{\alpha}(\mu(s))}ds = \infty. \tag{3.3}$$

Now for all values of $t \geq t_1$ and $x \in M^{+}$, we have $x^{\alpha}(\mu(t)) \geq x^{\alpha}(\mu(t_1)) = C$,and consequently

$$\int_{t_1}^{t}\frac{z'(s)}{x^{\alpha}(\mu(s))}ds \leq \frac{1}{C}\int_{t_1}^{t} z'(s)ds = \frac{1}{C}\left(z(t) - z(t_1)\right).$$

From (3.3) we obtain

$$\lim_{t\to\infty} z(t) = \infty. \tag{3.4}$$

Since $z(t) = x(t) + p(t)x(\tau(t))$ and $x(t)$ is nondecreasing we have $z(t) \leq (1 + p(t))x(t)$.Thus from (3.4), by taking into account the boundaries of $p(t)$, we obtain $\lim\limits_{t\to\infty} x(t) = \infty$. This completes the proof. □

We conclude this paper with the following remarks.

Remark 3.1. In [11], the author considered equation (1.1) without maxima and established a Theorem 2.4 which implies WOS $= \phi$. In proving this theorem,the author assumes that when $x(t)$ is weakly oscillatory then $z(t)$ is also weakly oscillatory.However this assumption is wrong as the following example shows.

Let $x(t) = t + \sin(t)$ which is weakly oscillatory. But $z(t) = x(t) + x(t - \pi) = 2t - \pi$. Which is not weakly oscillatory. Hence the conclusion of the Theorem 2.4 in [11] is wrong.

Remark 3.2. In this paper we obtain sufficient conditions for the solutions of equation (1.1) to be oscilltory or weakly oscillatory. Further we establish conditions for the behavior of solutions in the class M^+ *and* M^-. Hence the paper complement and correct the results established in [11].

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