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# A METHOD OF INVERSION OF FOURIER TRANSFORMS ANS ITS APPLICATIONS

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## 1. INTRODUCTION

The problem of inversion of Fourier transforms is a frequently discussed topic in the theory of PDEs, Stochastic Processes and many other branches of Analysis. We consider here in more details an application of a method proposed in Financial Modeling. As a motivating example consider a frictionless market with no arbitrage opportunities and a constant riskless interest rate $r > 0$. Assuming the existence of a risk-neutral equivalent martingale measure $\mathbb{Q}$, we get the option value $V = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\varphi]$ at time 0 and maturity $T > 0$, where $\varphi$ is a reward function and the expectation $\mathbb{E}^{\mathbb{Q}}$ is taken with respect to the equivalent martingale measure $\mathbb{Q}$. Usually, the reward function $\varphi$ has a simple structure. Hence, the main problem is to approximate properly the respective density function and then to approximate $\mathbb{E}^{\mathbb{Q}}[\varphi]$. Here we offer an approximant for the density function without proof of any convergence results. These problems will be considered in details in our future publications.

## 2. THE RESULTS

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$, $\langle \mathbf{x},\mathbf{y}\rangle := \sum_{k=1}^{n} x_k y_k$ be the usual scalar product and $|\mathbf{x}| := \langle \mathbf{x},\mathbf{x}\rangle^{1/2}$. For $f(\mathbf{x}) \in L_1(\mathbb{R}^n)$ define its Fourier transform

$$\mathbf{F}f(\mathbf{y}) = \int_{\mathbb{R}^n} \exp\left(-i\left\langle \mathbf{x},\mathbf{y}\right\rangle\right) f(\mathbf{x}) d\mathbf{x}$$

and its formal inverse as

$$\left(\mathbf{F}^{-1}f\right)(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\left(i\left\langle \mathbf{x},\mathbf{y}\right\rangle\right) f(\mathbf{y}) d\mathbf{y}.$$

We will need the following well-known result (see e.g. [7]).

**Theorem 1.** *(Plancherel's theorem) The Fourier transform is a linear continuous operator from $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$. The inverse Fourier transform, $\mathbf{F}^{-1}$, can be obtained by letting*

$$\left(\mathbf{F}^{-1}g\right)(\mathbf{y}) = (2\pi)^{-n}\left(\mathbf{F}g\right)(-\mathbf{y})$$

*for any $g \in L_2(\mathbb{R}^n)$.*

The density function $p_t^{\mathbb{Q}}$ of any Lévy process $\mathbf{X} = \{\mathbf{X}_t\}_{t\in\mathbb{R}_+}$ can be expressed in terms of the characteristic function $\Phi^{\mathbb{Q}}(\mathbf{x},t) = \exp\left(-t\psi^{\mathbb{Q}}(\mathbf{y})\right)$ of the distribution of $\mathbf{X}$ as $p_t^{\mathbb{Q}} = (2\pi)^{-n}\mathbf{F}\left(\Phi^{\mathbb{Q}}(\mathbf{x},t)\right)$, where $\psi^{\mathbb{Q}}(\mathbf{y})$ is the characteristic exponent. According to the Khintchine-Lévy formula, for any Lévy process $\mathbf{X}$, the characteristic exponent $\psi$ admits the representation

$$\psi(\mathbf{y}) = \langle \mathbf{L}\mathbf{x},\mathbf{x}\rangle - i\langle \mathbf{h},\mathbf{x}\rangle - \int_{\mathbb{R}^n}\left(1-\exp\left(i\langle \mathbf{x},\mathbf{z}\rangle\right) - i\langle \mathbf{x},\mathbf{z}\rangle \chi_D(\mathbf{x})\right)\Pi(d\mathbf{x}) \qquad (1)$$

where $\chi_D(\mathbf{y})$ is the characteristic function of the unit ball in $\mathbb{R}^n$, $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{L}$ is a symmetric nonnegative-definite matrix and $\Pi(d\mathbf{y})$ is a measure such that

$$\int_{\mathbb{R}^n} \min\{1, \langle \mathbf{x},\mathbf{x}\rangle\}\,\Pi(d\mathbf{y}) < \infty, \Pi(\{\mathbf{0}\}) = 0.$$

See [6] for more details. For simplicity we assume absolute convergence of multiple series under consideration which is sufficient for our applications. Let $K : \mathbb{R}^n \to \mathbb{R}$ be a fixed kernel function and $\mathbf{A}$ be a nonsingular $n \times n$ matrix.

**Definition 2.** We say that $K \in \mathcal{K}$ if the series

$$\Upsilon^{-1}(\mathbf{y},\mathbf{z}) := \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathbf{F}(K)\left(\mathbf{z} + 2\pi\left(\mathbf{y}^{-1}\right)^T \mathbf{m}\right) \qquad (2)$$

converges absolutely and $\Upsilon(\mathbf{y},\mathbf{z})\,\mathbf{F}(K)(\mathbf{z}) \in L_1(\mathbb{R}^n)$.

The set $\mathcal{K}$ is sufficiently large for our applications. In particular, if $K(\mathbf{z}) > 0$, $\forall \mathbf{z} \in \mathbb{R}^n$ then instead of $\Upsilon(\mathbf{z}) \mathbf{F}(K)(\mathbf{z}) \in L_1(\mathbb{R}^n)$ we may just clime that $\mathbf{F}(K)(\mathbf{z}) \in L_1(\mathbb{R}^n)$. A typical example of $K \in \mathcal{K}$ is given by a Gaussian of the form $K(\mathbf{y}) = \exp\left(-|\mathbf{By}|^2\right)$, where $\mathbf{B}$ is a nonsingular matrix. In this case $K(\mathbf{y}) > 0$ and the condition (2) is easily verifiable. Fix a kernel function $K \in \mathcal{K}$. Consider the function

$$\widetilde{sk}(\mathbf{y}) := \frac{\det(\mathbf{A})}{(2\pi)^n} \int_{\mathbb{R}^n} \Upsilon(\mathbf{y}, \mathbf{z}) \mathbf{F}(K)(\mathbf{z}) \exp(i\langle \mathbf{y}, \mathbf{z}\rangle)\, d\mathbf{z}$$

which we call the *fundamental sk-spline*. It is possible to show that

$$\widetilde{sk}(\mathbf{Am}) = \begin{cases} 1, & m_k = 0, 1 \le k \le n, \\ 0, & \text{otherwise.} \end{cases} .$$

for any $\mathbf{m} \in \mathbb{Z}^n$ if $K \in \mathbf{K}$ [5] . The functions $\widetilde{sk}(\mathbf{y})$ are analogs of periodic fundamental splines introduced in [3, 4]

$$\tilde{sk}(x) = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{Re\lambda_j(x) Re\lambda_j(y) + Im\lambda_j(x) Im\lambda_j(y)}{|\lambda_j(y)|^2},$$

where $\lambda_j(y) = \sum_{\nu=1}^{n} \exp\left(\frac{2\pi i \nu j}{n}\right) K\left(y - \frac{2\pi\nu}{n}\right) \neq 0$, $1 \le j \le n-1$ and $K \in C\left(\mathbb{T}^1\right)$ is a fixed kernel function. Observe that

$$\tilde{sk}(y + 2\pi j/n) = \begin{cases} 1, & j = 0 \mathrm{mod}(n), \\ 0 & \text{otherwise.} \end{cases}$$

Fundamental $sk$-splines on parallelepipedal grids in $\mathbb{T}^d$ (see [2]) were considered in [1]. To construct an approximant $q(\cdot)$ for the density function $p_t^{\mathbb{Q}}(\cdot)$ defined by $\Phi^{\mathbb{Q}}(\mathbf{x}, t)$ in (1) we assume that $K \in \mathcal{K} \cap L_2(\mathbb{R}^n)$. Consequently, by Plancherel theorem we get

$$\begin{aligned} p_t^{\mathbb{Q}}(\cdot) =& \frac{\mathbf{F}\left(\Phi^{\mathbb{Q}}(\mathbf{x}, t)\right)(\cdot)}{(2\pi)^n} \\ \approx& \mathbf{F}\left(\sum_{\mathbf{m}\in\mathbb{Z}^n} \frac{\Phi^{\mathbb{Q}}(\mathbf{Am},t)\widetilde{sk}(\mathbf{x} - \mathbf{Am})}{(2\pi)^n}\right)(\cdot) \\ =& \mathbf{F}\left(\sum_{\mathbf{m}\in\mathbb{Z}^n} \mathbf{F}^{-1}\left(\frac{\det(\mathbf{A})\Phi^{\mathbb{Q}}(\mathbf{Am},t)\mathbf{F}(K)(\mathbf{z})}{(2\pi)^n \sum_{\mathbf{s}\in\mathbb{Z}^n} \mathbf{F}(K)\left(\mathbf{z}+2\pi\left(\mathbf{y}^{-1}\right)^T \mathbf{s}\right)}\right)(\mathbf{x} - \mathbf{Am})\right)(\cdot) \\ =& \frac{\det(\mathbf{A})\mathbf{F}(K)(\cdot)\sum_{\mathbf{m}\in\mathbb{Z}^n} \Phi^{\mathbb{Q}}(-\mathbf{Am},t)\exp(i\langle \cdot, \mathbf{Am}\rangle)}{(2\pi)^n \sum_{\mathbf{s}\in\mathbb{Z}^n} \mathbf{F}(K)\left(\cdot + 2\pi\left(\mathbf{y}^{-1}\right)^T \mathbf{s}\right)}. \end{aligned}$$

Assume that for some domain $\Omega \subset \mathbb{R}^n$, $\Omega \ni \mathbf{0}$ the sum

$$\sum_{\mathbf{s}\in\mathbb{Z}^n\setminus\mathbf{0}} \mathbf{F}(K)\left(\cdot + 2\pi\left(\mathbf{y}^{-1}\right)^T \mathbf{s}\right)$$

is "relatively small" for any $\mathbf{z} \in \Omega$. Then

$$\sum_{\mathbf{s}\in\mathbb{Z}^n} \mathbf{F}(K)\left(\cdot + 2\pi \left(\mathbf{y}^{-1}\right)^T \mathbf{s}\right) \approx \mathbf{F}(K)(\cdot).$$

Hence

$$p_t^{\mathbb{Q}}(\cdot) \approx q(\cdot) = \frac{\det(\mathbf{A})}{(2\pi)^n} \sum_{\mathbf{m}\in\mathbb{Z}^n} \Phi^{\mathbb{Q}}(-\mathbf{Am},t) \exp\left(i\langle \cdot, \mathbf{Am}\rangle\right).$$

Let $\mathbf{m}_k$, $k \in \mathbb{N}$ corresponds to the nonincreasing rearrangement of $\left|\Phi^{\mathbb{Q}}(-\mathbf{Am},t)\right|$, $\mathbf{m} \in \mathbb{Z}^n$. Hence for a fixed $N = N\left(\Phi^{\mathbb{Q}}, \mathbf{y}, t,\right)$ we get

**Theorem 3.** *In our notations the approximant $q(\mathbf{z})$ for the density function $p_t^{\mathbb{Q}}(\mathbf{z})$ has the form*

$$q(\mathbf{z}) = \frac{\det(\mathbf{A})}{(2\pi)^n} \sum_{k=1}^{N} \Phi^{\mathbb{Q}}(-\mathbf{Am}_k, t) \exp\left(i\langle \mathbf{z}, \mathbf{Am}_k\rangle\right).$$

**Example 4.** Let $n = 2$ and $p(\mathbf{y}) = p(x_1, x_2) = \pi^{-1}\exp\left(-x_1^2 - x_2^2\right)$ be Gaussian density. Then $\Phi(\mathbf{y}) = \mathbf{F}p(-\mathbf{y}) = \exp\left(-\left(x_1^2 + x_2^2\right)/4\right)$. Let $P$ and $M$ be fixed parameters. In the case of the square grid $(2\pi k/P, 2\pi s/P)$, $(k, s) \in \mathbb{Z}^2$ we get $\mathbf{y} = \operatorname{diag}(2\pi/P, 2\pi/P)$, $\det(\mathbf{y}) = (2\pi/P)^2$. Hence the approximant $q(\mathbf{y})$ takes the form

$$\begin{aligned} q(\mathbf{y}) =& q(x_1, x_2) \\ =& \frac{(2\pi/P)^2}{(2\pi)^2} \sum_{|k|\le M} \sum_{|s|\le M} \Phi\left(-\frac{2\pi k}{P}, -\frac{2\pi s}{P}\right) \exp\left(\frac{2\pi k i}{P} x_1 + \frac{2\pi s i}{P} x_2\right) \\ =& \frac{1}{P^2} \sum_{|k|\le M} \sum_{|s|\le M} \exp\left(-\left(\frac{2\pi}{P}\right)^2 \left(\frac{k^2+s^2}{4}\right)\right) \exp\left(\frac{2\pi i k x_1}{P} + \frac{2\pi i s x_2}{P}\right). \end{aligned}$$

Let $d(P, M, a) := \max\left\{\mathbf{x} \in [-a/2, a/2] \times [-a/2, a/2] \mid |p(\mathbf{y}) - q(\mathbf{y})|\right\}$. Numerical examples show that $d(5, 4, 1) = 2.36 \times 10^{-5}$, $d(5, 6, 1) = 1.8 \times 10^{-8}$.

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