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A FAMILY OF RECURRENCE GENERATED FUNCTIONS BASED ON THE LOGISTIC FUNCTION WITH POLYNOMIAL VARIABLE TRANSFER. SOME APPROXIMATION AND MODELLING ASPECTS

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ABSTRACT: In this note we construct a family of recurrence generated sigmoidal functions based on the Verhulst logistic function with polynomial variable transfer.

We prove estimates for the Hausdorff approximation of the Heaviside step function by means of this family. Numerical examples, illustrating our results are given.

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Key Words: sigmoidal functions, Verhulst and Pearl logistic models, Recurrence generated family, Heaviside function, Hausdorff distance, upper and lower bounds

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1. INTRODUCTION

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

The logistic function $s(t) = \frac{a}{1+e^{-kt}}$ was introduced by Pierre François Verhulst [1]–[3], who applied it to human population dynamics. Verhulst proposed his logistic equation to describe the mechanism of the self-limiting growth of a biological population.

Studying "Canteloup growth", Pearl et al. [4]–[5] empirically found that one should generalized the logistic map in order to reproduce better the data.

Rather than the mere logistic, they propose a form like $y(t) = \frac{r}{1+e^{a_0+a_1t+a_2t^2+\ldots}}$, where y, is the number of seedlings of the canteloups.

An extensive overview on the topic can be found in [6], see also [7]–[8]).

In [9] we propose a new class of growth curves, generated by reaction networks, based on the insertion of "correcting amendments" of polynomial–type: *"Verhulst curve of growth with polynomial variable transfer"* with application to approximate some "specific data".

Definition 1. *The (basic) step function is:*

$$h_0(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ 1/2, & \text{if} \quad t = 0, \\ 1, & \text{if} \quad t > 0, \end{cases}$$

usually known as Heaviside step function.

Definition 2. *[11] The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{1}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

More precise bounds for the H–distance d between the Heaviside step function h_0 and the sigmoid Verhulst function can be found in [7].

A family of recurrence generated functions based on the Verhulst "basic" logistic function can be found in [10].

In this paper we consider a family of recurrence generated sigmoidal functions based on the Verhulst logistic function with polynomial variable transfer.

Estimates for the Hausdorff approximation of the Heaviside step function by means of this family are also given.

2. MAIN RESULTS

Let us consider the following family of recurrence generated sigmoidal logistic functions

$$M_{i+1}(t) = \frac{1}{1 + k_{i+1}e^{-F(t)-M_i(t)}}, \quad i = 0, 1, 2, \ldots, \tag{2}$$

with

$$M_{i+1}(0) = \frac{1}{2}, \quad i = 0, 1, 2, \ldots, \tag{3}$$

based on the Verhulst logistic function with "polynomial variable transfer"

$$M_0(t) = \frac{1}{1+e^{-F(t)}},$$
$$F(t) = \sum_{i=0}^{n} a_i t^i, \;\; a_0 = 0. \tag{4}$$

From (3) we have $k_{i+1} = e^{\frac{1}{2}}$ for $i = 0, 1, 2, \ldots,$.

Denote the number of recurrences by p.

The H-distance d_p between the Heaviside step function h_0 and the sigmoidal family satisfies the relation:

$$H_p(d_p) := M_p(d_p) - 1 + d_p = 0. \tag{5}$$

Special case $n = 2$**.**

Theorem 1. *For given p, the H-distance d_p between the function h_0 and the function M_p can be expressed in terms of the rate parameter a_1 for any $a_1 \geq e$ as follows:*

$$d_{l_p} = \frac{1}{4\left(1 + \frac{a_1}{2^{2(p+1)}}\sum_{i=0}^{p} 2^{2i}\right)} < d_p < \frac{\ln\left(4\left(1 + \frac{a_1}{2^{2(p+1)}}\sum_{i=0}^{p} 2^{2i}\right)\right)}{4\left(1 + \frac{a_1}{2^{2(p+1)}}\sum_{i=0}^{p} 2^{2i}\right)} = d_{r_p}. \tag{6}$$

Proof. We note that the function

$$G_p(d_p) = -\frac{1}{2} + \left(1 + \frac{a_1}{2^{2(p+1)}}\sum_{i=0}^{p} 2^{2i}\right) d_p.$$

approximates $H_p(d_p)$ with $d_p \to 0$ as $O(d_p^2)$ (see, for example Fig. 3a for $p = 1$).

In addition $G_p'(d_p) > 0$ and for $a_1 \geq e$

$$G_p(d_{l_p}) < 0; \quad G_p(d_{r_p}) > 0.$$

This completes the proof of the inequalities (6).

The question of finding precise two-sided estimates for the magnitude of the Hausdorff approximation of the Heaviside function with classes of the indicated family $M_i(t)$ remains open.

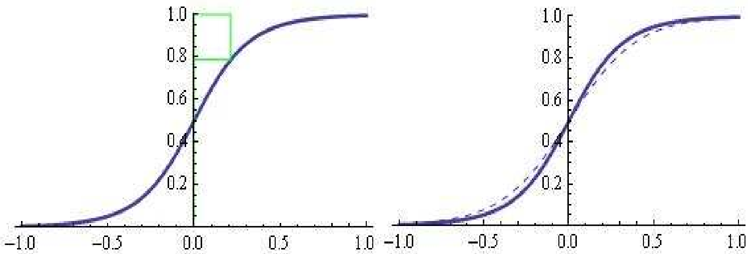

Figure 1: Comparison between $M_1(t)$ - (thick) and Verhulst's model $v(t) = 1/(1+e^{-a_1 t})$ - (dashed) at fixed $n = 2$, $a_1 = 5$, $a_2 = -0.01$; Hausdorff distance $d_1 = 0.212902$, $d_{l_1} = 0.097561$, $d_{r_1} = 0.227051$.

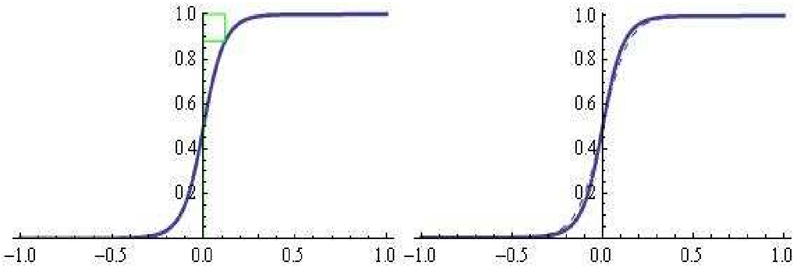

Figure 2: Comparison between $M_1(t)$ - (thick) and Verhulst's model $v(t) = 1/(1+e^{-a_1 t})$ - (dashed) at fixed $n = 2$, $a_1 = 14$, $a_2 = -0.05$; Hausdorff distance $d_1 = 0.118831$, $d_{l_1} = 0.0465116$, $d_{r_1} = 0.1427$.

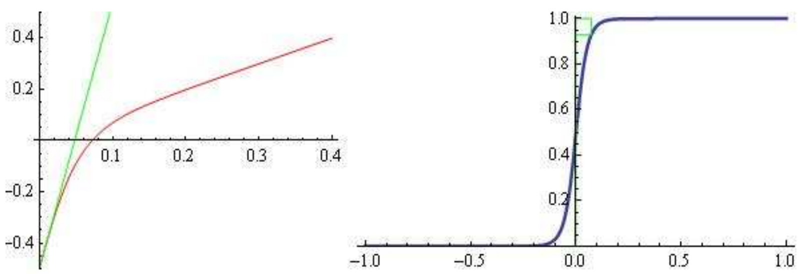

Figure 3: a) The functions $H_1(d_1$ and $G_1(d_1)$; b) Comparison between $M_1(t)$ - (thick) and Verhulst's model $v(t) = 1/(1+e^{-a_1 t})$ - (dashed) at fixed $n = 2$, $a_1 = 30$, $a_2 = -0.5$; Hausdorff distance $d_1 = 0.0720545$, $d_{l_1} = 0.0240964$, $d_{r_1} = 0.0897754$.

The task is greatly complicated by the intrinsic properties of the generated class of

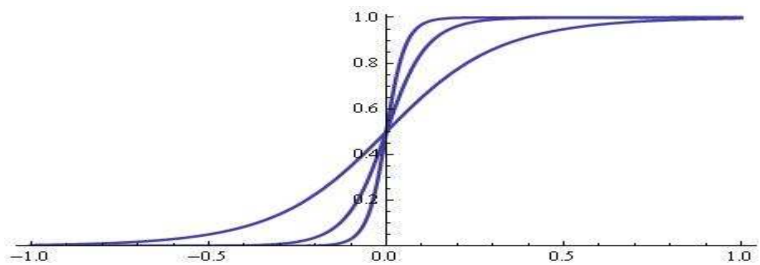

Figure 4: A typical recurrent embedding of the series: $M_0(t)$, $M_1(t)$ and $M_2(t)$.

Day No.	Faults	Cumulative faults	Day No.	Faults	Cumulative faults	Day No.	Faults	Cumulative faults	Day No.	Faults	Cumulative faults
1	2	2	20	6	124	39	5	204	58	3	303
2	0	2	21	2	126	40	1	205	59	4	307
3	2	4	22	3	129	41	4	209	60	8	315
4	3	7	23	2	131	42	6	217	61	3	318
5	3	10	24	3	134	43	3	220	62	4	322
6	6	16	25	8	142	44	2	222	63	5	327
7	8	24	26	6	148	45	6	228	64	6	333
8	8	32	27	7	155	46	13	241	65	0	333
9	12	44	28	8	163	47	9	250	66	4	337
10	10	54	29	2	165	48	6	256	67	5	342
11	6	60	30	3	168	49	7	263	68	4	346
12	5	65	31	4	172	50	3	266	69	5	351
13	4	69	32	3	175	51	3	269	70	5	356
14	6	75	33	3	178	52	4	273	71	5	361
15	10	85	34	4	182	53	5	278	72	3	364
16	6	91	35	4	186	54	6	284	73	3	367
17	7	98	36	5	191	55	6	290			
18	10	108	37	4	195	56	5	295			
19	10	118	38	4	199	57	5	300			

Figure 5: "LP1–Data Set" [26], [27].

functions, as well as by the type and location of zeros of polynomial $F(t)$ [12].

Some comparisons between Verhulst's model $s_0(t)$ and the model (2) are visualized on Fig. 1 – Fig. 4.

3. NUMERICAL EXPERIMENT

We consider the following real–world reliability data set (LP1–Data Set) reported by [26].

This data set was collected from a military control system by system testing and testing on board.

The Fig. 5 presents the details of the failure data and the time unit is day. There are totally 367 faults detected within 73 days (see, [27].

The fitting results by 17 Models for the "LP1–Data Set" are given in [27] (see, Fig. 6).

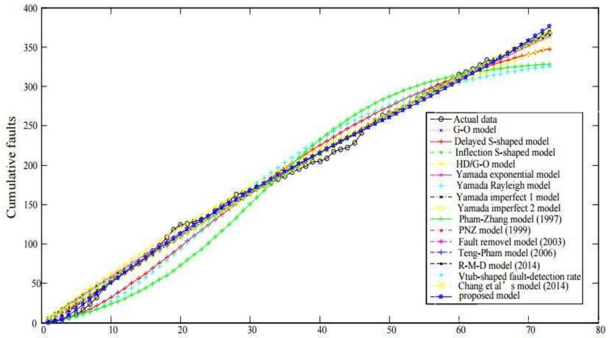

Figure 6: The fitting results by 17 models (G-O model, Delayed S–shaped, Inflection S–shaped, $HD/G-O$, Yamada exponential, Yamada Rayleigh, Yamada imperfect, Pham–Zhang (1997), PNZ (1999), Fault removal model (2003), Teng–Pham (2006), R–M–D (2014), Vtub–shaped fault–detection rate (2014), Chang (2014), Li–Pham (2019)) for the "LP1–Data Set"(see, for details [27]).

For the "LP1–Data Set" and fixed $p = 1$, the model - (2) for

$$n = 7,\ A = 368,\ a_0 = 0,\ a_1 = -10.6027,\ a_2 = 11.6604,\ a_3 = -5.37286,$$

$$a_4 = 1.30612,\ a_5 = -0.174948,\ a_6 = 0.0122102,\ a_7 = -0.000345938$$

is depicted on Fig. 7. (We have adopted a scale on the horizontal axis: 0.1 division corresponds to 1 time interval).

4. CONCLUDING REMARKS

Constructive approximation theory by superposition of sigmoidal functions can be found in [13]. For contemporary applicable study on sigmoids and some of their applications see [8], [14], [15]–[25].

The reader may formulate the corresponding approximation problem for arbitrary n following the ideas given in this note, and will be omitted.

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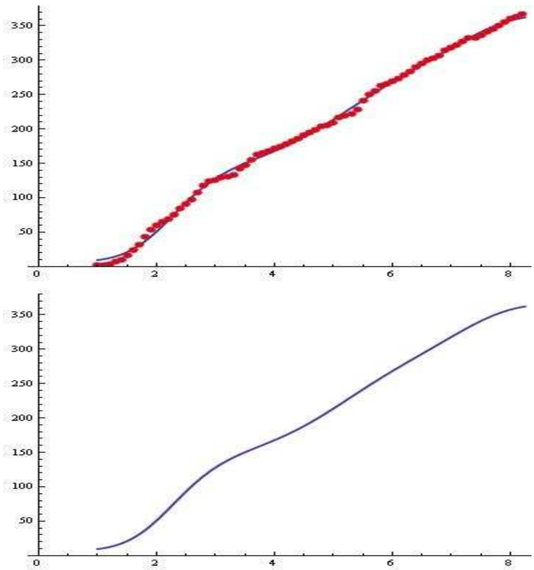

Figure 7: a) The "LP1–Data Set"; b) The model (2) ($n = 7$, $A = 368$, $a_0 = 0$, $a_1 = -10.6027$, $a_2 = 11.6604$, $a_3 = -5.37286$, $a_4 = 1.30612$, $a_5 = -0.174948$, $a_6 = 0.0122102$, $a_7 = -0.000345938$) for the "LP1–Data Set" [26], [27].

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