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A LOOK AT THE INVERSE WEIBULL "ADAPTIVE FUNCTION": PROPERTIES AND APPLICATIONS

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ABSTRACT: In this note we study some properties of an new inverse Weibull cumulative function proposed by Afify, Shawky and Nassar [1]. More precisely, we prove estimates for the "saturation" - d about Hausdorff metric. A new activation functions are defined. We consider also modified families of functions with "polynomial variable transfer" with applications to the Antenna–feeder Analysis. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

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Key Words: inverse Weibull cumulative function, Hausdorff distance, adaptive function, modified families of functions with "polynomial variable transfer"

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1. INTRODUCTION

The Weibull distribution has been widely used in survival and reliability analyses. Some modifications, properties and applications of Weibull and Weibull-R families of distributions can be found in [2]–[13].

Definition 1. *In [1] the authors proposed the following new probability distribution with cumulative distribution function:*

$$M(t) = 1 - \frac{\ln\left(1+\delta-\delta^{e^{-\lambda t^{-\alpha}}}\right)}{\ln\delta} \tag{1}$$

for $t > 0,\ \alpha > 0,\ \lambda > 0,\ \delta > 0.$

Various modifications of this "powerful" class of functions have been proposed and studied by a number of researchers.

We consider the following one–parameter family:

$$F(t) = 1 - \frac{\ln\left(1+\delta-\delta^{e^{-\frac{\delta}{t}}}\right)}{\ln\delta}. \tag{2}$$

Definition 2. *[14], [15] The Hausdorff distance (the H–distance) [14]* $\rho(f,g)$ *between two interval functions* f, g *on* $\Omega \subseteq \mathbb{R}$*, is the distance between their completed graphs* $F(f)$ *and* $F(g)$ *considered as closed subsets of* $\Omega \times \mathbb{R}$*. More precisely,*

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{3}$$

wherein $||.||$ *is any norm in* $\mathbb{R}^2$*, e. g. the maximum norm* $||(t,x)|| = \max\{|t|,|x|\}$*; hence the distance between the points* $A = (t_A, x_A)$*,* $B = (t_B, x_B)$ *in* $\mathbb{R}^2$ *is* $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$*.*

In this article we study some properties of the family (2) and prove estimates for the "saturation" - d about Hausdorff metric.

2. MAIN RESULTS

1. For the "saturation" - d in the Hausdorff sense to the horizontal asymptote using $F(t)$ we have

$$F(d) = 1 - d, \tag{4}$$

i.e. d is the solution of the nonlinear equation

$$e^{\frac{\delta}{d}} - \frac{\ln\delta}{\ln\left(1+\delta-\delta^d\right)} = 0.$$

Let

$$c(\delta,d) = \frac{-\delta^d}{\ln\left(1+\delta-\delta^d\right)} := c.$$

For $0 < \delta,\ d \le \frac{1}{2}$, we see that

$$\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \le c(\delta,d) \le 1.$$

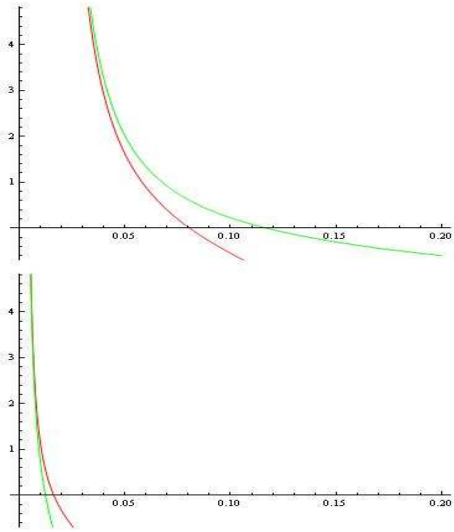

Figure 1: The functions $G(d)$ and $H(d)$ for: a) $\delta = 0.06$; b) $\delta = 0.01$.

Let

$$G(d) = e^{\frac{\delta}{d}} + c\frac{\ln\delta}{\delta^d} = 0. \tag{5}$$

Consider the following good approximation to the $G(t)$ (see, Fig. 1).

$$H(d) = e^{\frac{\delta}{d}+d} - c\ln\frac{1}{\delta} = 0. \tag{6}$$

From (6) we find

$$\frac{\delta}{d} + d = \ln c + \ln\left(\ln\left(\frac{1}{\delta}\right)\right) := 2b$$

$$d = \frac{\delta}{b+\sqrt{b^2-\delta}}.$$

For sufficiently small values of δ we have

$$d \approx \frac{\delta}{2b}.$$

Thus, we prove the following

Theorem 1. Let $0 < \delta$, $d \leq \frac{1}{2}$. For the "saturation" - d we have

$$d = \frac{\delta}{b+\sqrt{b^2-\delta}}. \tag{7}$$

For sufficiently small values of δ we have

$$d \approx \frac{\delta}{2b} \approx \frac{\delta}{1+\ln\left(\ln\left(\frac{1}{\delta}\right)\right)}. \tag{8}$$

The modified family $F(t)$ for

a) $\delta = 0.12,\ d = 0.148095;$

b) $\delta = 0.06,\ d = 0.0805819;$

c) $\delta = 0.01,\ d = 0.0163254$

is plotted on Fig. 2.

2. Let t_0 is the value for which $F(t_0) = \frac{1}{2}$.

The one–sided Hausdorff distance d_1 between $F(t)$ and the shifted Heaviside step function

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

satisfies the relation:

$$F(t_0 + d_1) = 1 - d_1. \tag{9}$$

The following theorem gives upper and lower bounds for d_1.

Theorem 2. Let $\delta \in (0.06, 0.5)$ and $B > \frac{1}{2.1}e^{1.05}$ where

$$B = 1 + (1 + \delta - \sqrt{\delta})\delta^{-\frac{3}{2}} \frac{\ln(1 + \delta - \sqrt{\delta})}{\ln \delta} \ln^2 \left(\frac{\ln(1 + \delta - \sqrt{\delta})}{\ln \delta} \right)$$

Then the one–sided Hausdorff distance d_1 between $F(t)$ and $h_{t_0}(t)$ satisfies the following inequalities

$$d_{1,l} := \frac{1}{2.1B} < d_1 < \frac{\ln 2.1B}{2.1B} := d_{1,r}. \tag{10}$$

The Proof follows the ideas given in [16] and will be omitted.

Some computational examples using relations (9)–(10) are presented in Table 1.

δ	t_0	$d_{1,l}$	d_1 *computed by* (9)	$d_{1,r}$
0.07	0.0279024	0.020462	0.0770817	0.0795805
0.08	0.0331862	0.0245766	0.0859485	0.09108
0.1	0.0445214	0.0334624	0.102866	0.113683
0.15	0.0770659	0.0587756	0.14125	0.166572
0.2	0.11529	0.0870713	0.175118	0.212543
0.25	0.158979	0.116742	0.205259	0.250737
0.3	0.2008027	0.146471	0.232184	0.28136

Table 1: Bounds for d_1 computed by (9)–(10) for various values of δ.

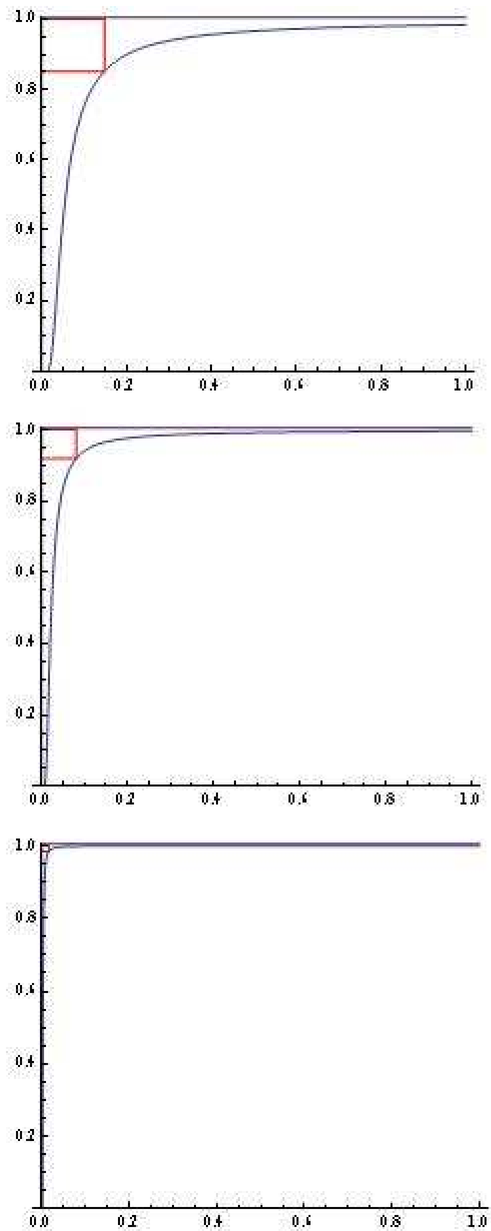

Figure 2: The function F for: a) $\delta = 0.12$, $d = 0.148095$; b) $\delta = 0.06$, $d = 0.0805819$; c) $\delta = 0.01$, $d = 0.0163254$.

3. SOME APPLICATIONS

1. Consider the following model with "polynomial variable transfer":

$$F^*(t) = 1 - \frac{\ln\left(1+\delta-\delta^{e^{-\frac{\delta}{f(t)}}}\right)}{\ln \delta},$$

$$f(t) = \sum_{i}^{n} a_i t^i, \quad a_0 = 0. \tag{11}$$

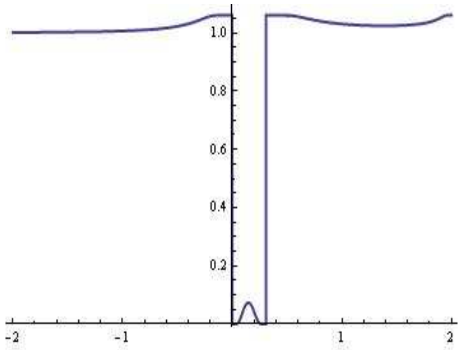

Figure 3: A typical filter characteristic by using model $F^*(t)$ for $n = 3$, $\delta = 0.13$, $a_0 = 0$, $a_1 = 0.4$, $a_2 = -1.5$, $a_3 = 0.65$.

For example, a typical filter characteristic by using model $F^*(t)$ for $n = 3$, $\delta = 0.13$, $a_0 = 0$, $a_1 = 0.4$, $a_2 = -1.5$, $a_3 = 0.65$ is plotted on Fig. 3.

2. Consider the following adaptive functions:

$$M_1(t) = 1 - \frac{\ln\left(1 + \delta - \delta^{e^{-\frac{\delta}{|t(1-t)(2-t)|}}}\right)}{\ln\delta}. \tag{12}$$

$$M_2(t) = 1 - \frac{\ln\left(1 + \delta - \delta^{e^{-\frac{\delta}{|t(1-t)(2-t)(3-t)|}}}\right)}{\ln\delta}. \tag{13}$$

Obviously, these new adaptive functions ($M_1(t)$ and $M_2(t)$) can be used successfully to simulate typical filter characteristics.

The models $M_1(t)$ for $\delta = 1.31$ and $M_2(t)$ for $\delta = 1.91$ are plotted on Fig. 4.

3. Let $t = b\cos\theta + c$. Consider the function $|F^*(t)|$.

Then, for example, typical emitting charts using $|F^*(t)|$ for

a) $n = 4$, $\delta = 0.3$, $a_0 = 0$, $a_1 = 0.99$, $a_2 = -3.8$, $a_3 = 5.7$, $a_4 = -4.8$, $b = 4.5$, $c = -0.18$;

b) $n = 3$, $\delta = 0.22$, $a_0 = 0$, $a_1 = -0.01$, $a_2 = 1.1$, $a_3 = -1.1$, $b = 3.4$, $c = -0.18$ are plotted on Fig. 5 – Fig. 6.

A typical emitting chart using $|F^*(\theta)|$ for $n = 6$, $\delta = 0.22$, $a_0 = 0$, $a_1 = -0.01$, $a_2 = 1.1$, $a_3 = -1.1$, $a_4 = 0.15$, $a_5 = 0.5$, $a_6 = -0.02$ $b = 1.2$, $c = 0.005$ is depicted on Fig. 7.

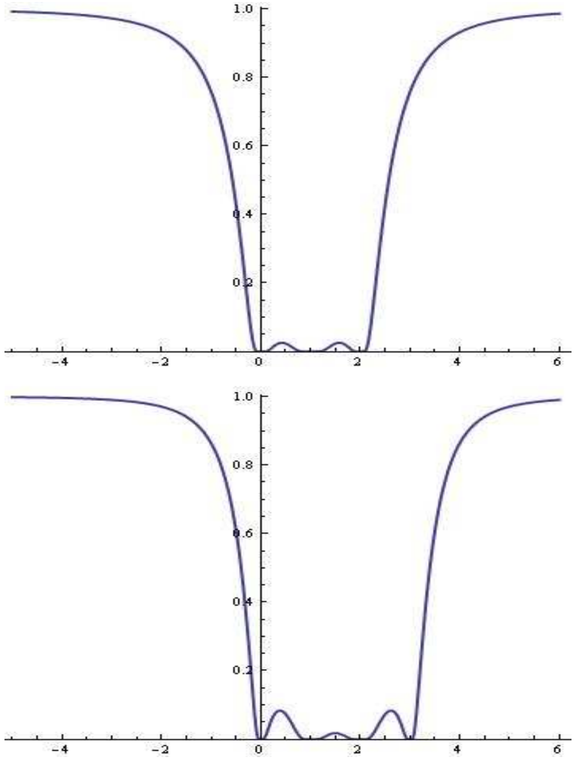

Figure 4: The models $M_1(t)$ for $\delta = 1.31$ and $M_2(t)$ for $\delta = 1.91$ as a typical filter characteristic.

Consider the following modification of the model (1):

$$H(t) = 1 - \frac{\ln\left(1 + \delta - \delta^{e^{-\frac{\delta}{t^\delta}}}\right)}{\ln \delta}. \qquad (14)$$

For the "saturation" - d_2 in the Hausdorff sense to the horizontal asymptote using $H(t)$ we have

$$H(d_2) = 1 - d_2, \qquad (15)$$

i.e. d_2 is the solution of the nonlinear equation

$$e^{\frac{\delta}{d_2^\delta}} - \frac{\ln \delta}{\ln\left(1 + \delta - \delta^d\right)} = 0.$$

The reader can formulate the corresponding approximation problem for this cumulative function following results from Theorem 1.

The basic problems considered in [17] (see, also [18]–[28]) are approximation of functions and point sets by algebraic and trigonometric polynomials in Hausdorff metric as well as their applications in the field of antenna-feeder technique, analysis and

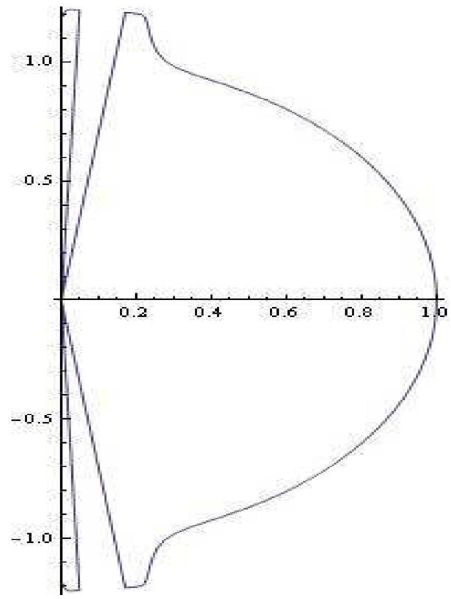

Figure 5: A typical emitting chart using $|F^*(t)|$ for $n = 4$, $\delta = 0.3$, $a_0 = 0$, $a_1 = 0.99$, $a_2 = -3.8$, $a_3 = 5.7$, $a_4 = -4.8$, $b = 4.5$, $c = -0.18$.

synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.

Unfortunately, these diagrams cannot always be realized in practice.

Specialists working in these scientific fields have a say.

For some modelling and approximation problems, see [29]–[49].

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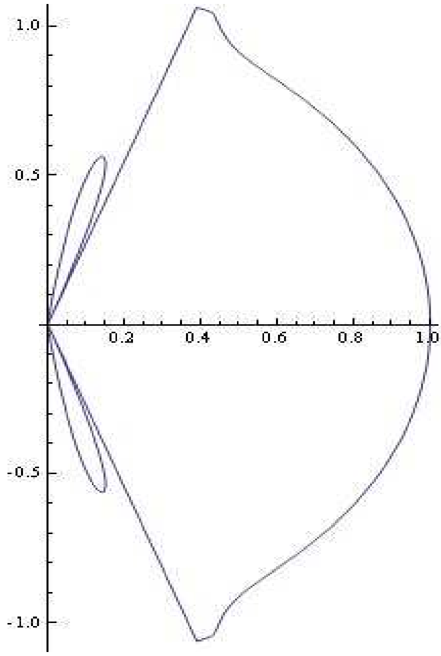

Figure 6: A typical emitting chart using $|F^*(t)|$ for $n = 3$, $\delta = 0.22$, $a_0 = 0$, $a_1 = -0.01$, $a_2 = 1.1$, $a_3 = -1.1$, $b = 3.4$, $c = -0.18$.

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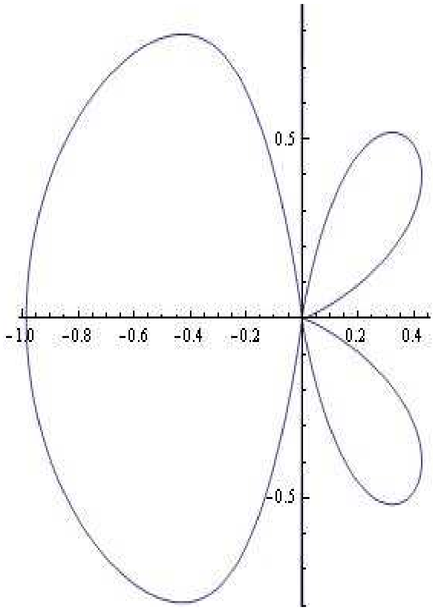

Figure 7: A typical emitting chart using $|F^*(\theta)|$ for $n = 6$, $\delta = 0.22$, $a_0 = 0$, $a_1 = -0.01$, $a_2 = 1.1$, $a_3 = -1.1$, $a_4 = 0.15$, $a_5 = 0.5$, $a_6 = -0.02$ $b = 1.2$, $c = 0.005$.

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