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ON THE APPROXIMATION OF THE HAAR SCALING FUNCTION BY SIGMOIDAL SCALING FUNCTIONS

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ABSTRACT: In this article we study the Hausdorff approximation of the Haar scaling function by sigmoidal scaling functions. We prove upper and lower estimates for the value of the Hausdorff distance d. A simple dynamic software module using CAS Mathematica and Wolfram Cloud Open Access is developed. Numerical examples are given to illustrate our results.

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Key Words: Haar scaling function, sigmoidal scaling function, Hausdorff distance, two–sided bounds

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1. INTRODUCTION

From the perspective of applied mathematics and modeling sigmoid functions find their place in numerous areas of life and social sciences, physics and engineering, to mention a few familiar applications: population dynamics, artificial neural networks, signal and image processing antenna–feeder technique, finances and insurance.

Definition 1. *For $r \in \mathbb{R}$ denote by $h_r \in \mathbb{H}(\mathbb{R})$ the (interval) Heaviside step function given by*

$$h_r(t) = \begin{cases} 0, & \text{if} \quad t < r, \\ [0,1], & \text{if} \quad t = r, \\ 1, & \text{if} \quad t > r. \end{cases}$$

For $r = 0$ we obtain the basic Heaviside step function $h_0(t)$.

Definition 2. *Sums of sigmoid functions. For a given vector $r = (r_1, r_2, ..., r_k) \in \mathbb{R}^k$, such that $r_1 < r_2 < ... < r_k$, and a vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{R}^k$ denote*

$$H(r, \alpha; t) = \sum_{i=1}^{k} \alpha_i h_{r_i}(t).$$

Function $H(r, \alpha; t)$ is a step function with k steps (jumps).

Several practically important families of smooth sigmoid functions arise from population dynamics.

Sigmoid functions find multiple applications to neural networks and cell growth population models. A classical example is the Verhulst population growth model. Verhulst model makes an extensive use of the *logistic* sigmoid function

$$s_0(t) = \frac{a_1}{1 + e^{-kt}}.$$

Definition 3 ([6, 7])**.** *The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \}, \tag{1}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is
$||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Theorem 4 ([8])**.** *For the Hausdorff distance $d = \rho(h_0, s_0)$ between the Heaviside step function h_0 and the sigmoid Verhulst function s_0 (with $a_1 = 1$) the following inequalities hold for $k \geq 2$:*

$$d_l = \frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{(k+1)\left(1 + \frac{1}{\ln(k+1)}\right)} < d < \frac{\ln(k+1)}{k+1} + \frac{\ln\ln(k+1)}{(k+1)\left(\frac{\ln\ln(k+1)}{1-\ln(k+1)} - 1\right)} = d_r. \tag{2}$$

The Verhulst logistic model is considered as a basic example to introduce several related mathematical problems: approximation of step and cut functions by means of logistic function, fitting a sigmoid model to time course measurement data, etc.

Similarly, one can construct sums of other suitably shifted sigmoid functions.

Here we are interested in arbitrary shifted (horizontally translated) logistic functions.

Both the step function and the logistic function preserve their form under horizontal translation—note that Verhulst equation possess constant isoclines.

Focusing on the shifted logistic function we have

$$s_r(t) = s_0(t-r) = \frac{a}{1+e^{-k(t-r)}}.$$

The superposition of sigmoid functions is given by:

$$S(t) = \sum_{i=1}^{n} \frac{a_i}{1+e^{-k_i(t-r_i)}}. \tag{3}$$

Definition 5 ([1, 2])**.** *The sigmoidal scaling function is defined by:*

$$\psi_a(t) = \frac{1}{1+e^{-at}} - \frac{1}{1+e^{-a(t-1)}} \tag{4}$$

where the parameter a controls the transition between plateaus, which may be smooth or sharp if a is small or large, as it is shown on Figure 1.

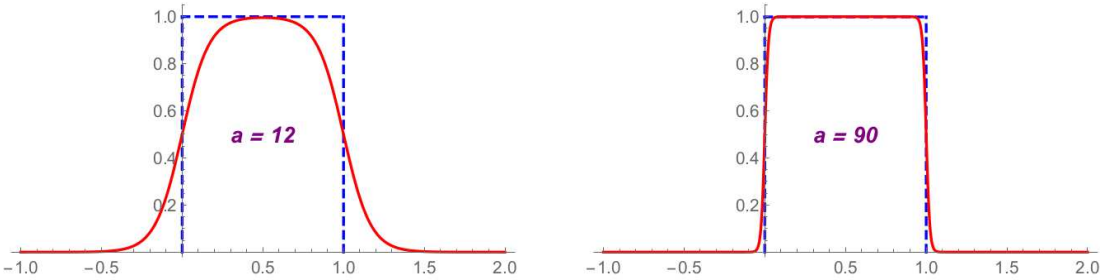


Figure 1: Sigmoidal scaling function $\psi_a(t)$ for $a = 12$ and $a = 90$.

Definition 6. *The Haar scaling function is defined by:*

$$\psi_H(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Noting that $\psi_a(t)$ resembles the *Haar scaling function.*

In this article we study the Hausdorff approximation [6, 7] of Haar scaling function $\psi_H(t)$ by sigmoidal scaling function $\psi_a(t)$.

2. MAIN RESULTS

2.1. APPROXIMATION RESULTS

In this section we investigate the value of Hausdorff distance d between Haar scaling function $\psi_H(t)$ and sigmoidal scaling function $\psi_a(t)$ given by (5) and (4), respectively. Then the following equation

$$\psi_a(d) = 1 - d.$$

holds true. Let us consider the function

$$F(d) = \psi_a(d) - 1 + d. \tag{6}$$

Now we examine the following approximation to the $F(d)$

$$G(d) = A + Bd,$$

where A and B are defined by

$$A = -\frac{1}{2} - \frac{1}{1+e^a} \quad \text{and} \quad B = 1 + \frac{a}{4} - \frac{ae^a}{(1+e^a)^2}. \tag{7}$$

Evidently, $G(d)$ approximates $F(d)$ with $\mathcal{O}(d^2)$ as $d \to 0$ (see, Figure 2).

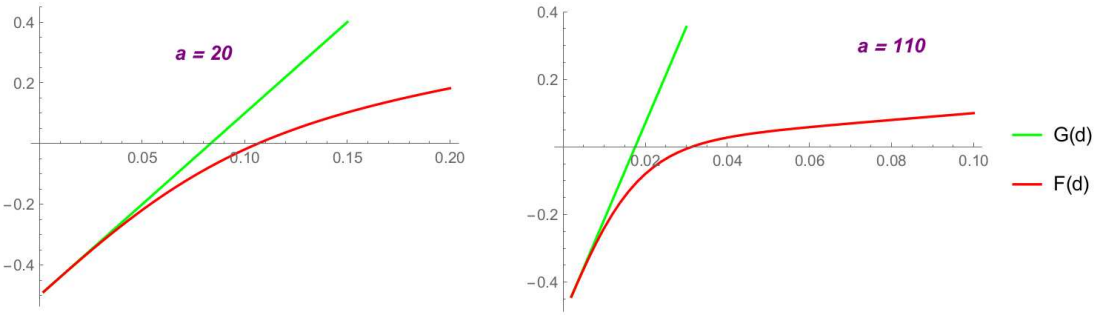


Figure 2: The functions $G(d)$ and $F(d)$ for $a = 20$ and $a = 110$.

Let us define $d_l := \frac{1}{2B}$. Then it is easy to prove that

$$G(d_l) < 0.$$

If $d_r := \frac{\ln 2B}{2B}$, then for $a \geq 10$ we have

$$G(d_r) > 0.$$

Thus, we prove the following theorem that gives upper and lower estimates for the Hausdorff approximation d.

Theorem 7. *Let $a \geq 10$, $d \leq 0.2$. Then for the Hausdorff distance d between Haar scaling function $\psi_H(t)$ and sigmoidal scaling function $\psi_a(t)$ given by* (5) *and* (4)*, respectively we have*

$$d_l := \frac{1}{2B} < d < \frac{\ln 2B}{2B} := d_r, \tag{8}$$

where B is defined by (7)*.*

In Table 1 we present some computational examples for different values of parameter a. We use Theorem 7 for computation of values of upper and lower estimates d_l and d_r. As it can be seen the proven bottom estimates for the value of Hausdorff distance d is reliable in approximation of Haar scaling function $\psi_H(t)$ and sigmoidal scaling function $\psi_a(t)$. Several graphical representation are presented in Figure 3.

a	d_l	d computed by (6)	d_r
10	0.142876	0.163449	0.278005
20	0.0833333	0.106402	0.207076
60	0.03125	0.0493157	0.108304
110	0.0175439	0.0312254	0.0709307
150	0.012987	0.0245484	0.0564131
200	0.00980392	0.01957	0.0453429
300	0.00657895	0.0141467	0.0330518

Table 1: Bounds for d computed by (8) for various values of parameter a.

Using the previous conclusions we develop a simple dynamic programming module implemented within the programming environment *CAS Wolfram Mathematica* and *Wolfram Cloud Open Access* (see Figure 4). Using dynamical user-defined value of parameter a, the main options of the presented module are :

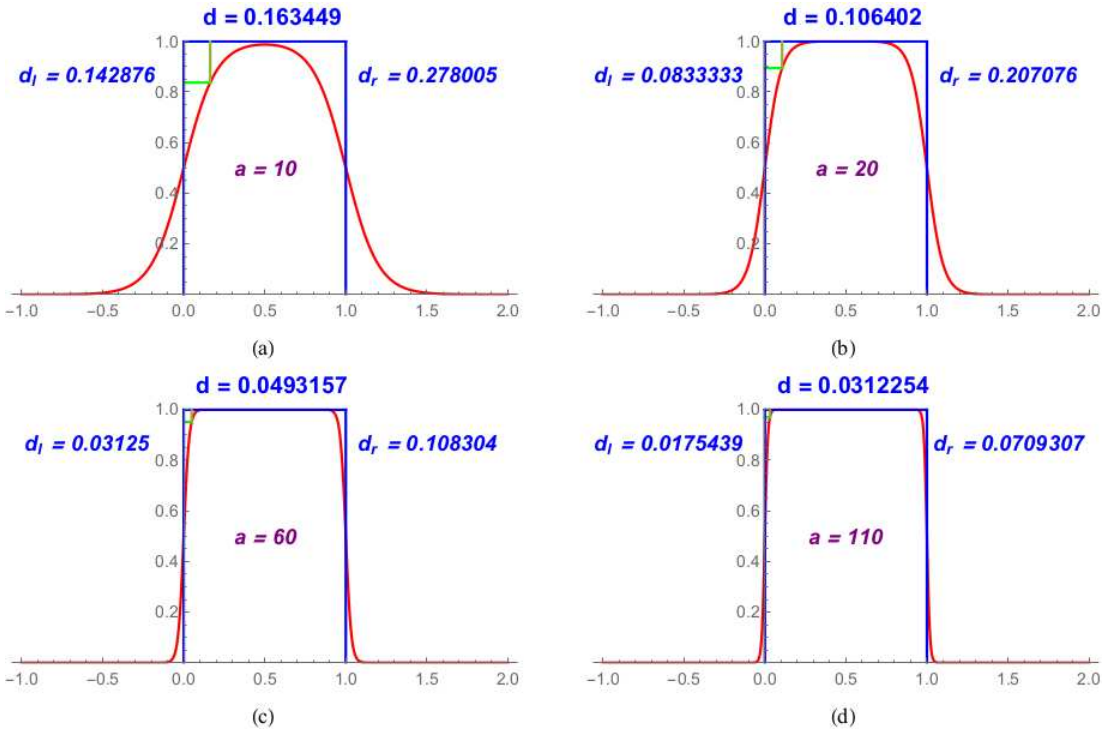


Figure 3: Approximation of $\psi_H(t)$ by $\psi_a(t)$ for different values of parameter a and corresponding Hausdorff distance d

- to compute the Hausdorff distance d between Haar scaling function $\psi_H(t)$ and sigmoidal scaling function $\psi_a(t)$;
- to visualize obtained results;
- to compute and visualize upper and lower estimates from Theorem 7;
- web (cloud) version of the module that requires only browser and internet connection.

2.2. A MODIFIED SCALING FUNCTION. APPLICATIONS

Consider the function

$$\overline{\psi}_a(t) = 2\left(\frac{1}{1+e^{-a|t|}} - \frac{1}{1+e^{-a|t-1|}}\right). \tag{9}$$

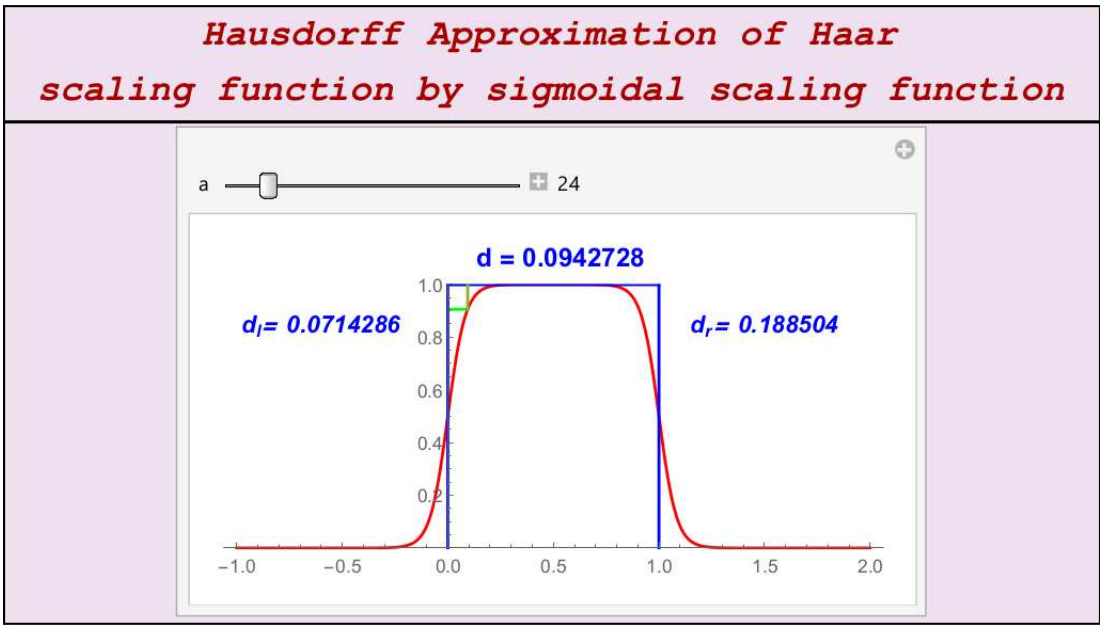


Figure 4: A simple module for computation and visualization of the Hausdorff distance between the Haar scaling function by sigmoidal scaling function.

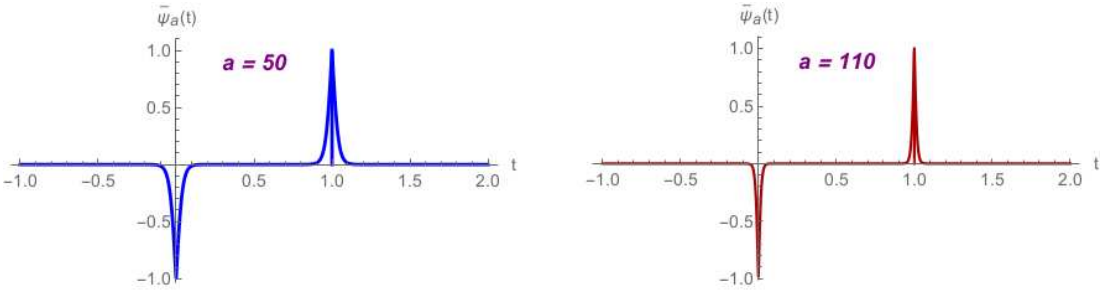


Figure 5: The modified family $\overline{\psi}_a(t)$

The problem of approximating the "point set" is also of independent interest (in the limiting case - the "point set CROSS"), see also the basic problem from paper [9]. In Figure 5 is depicted the modified family $\overline{\psi}_a(t)$, given in (9), for two cases: $a = 50$ and $a = 110$.

Let $t = r\cos\theta + c$, where θ is the azimuthal angle. Now we consider the function $\overline{\psi}_a(\theta)$. In Figure 6 we give an example for typical emitting chart using $\overline{\psi}_a(\theta)$.

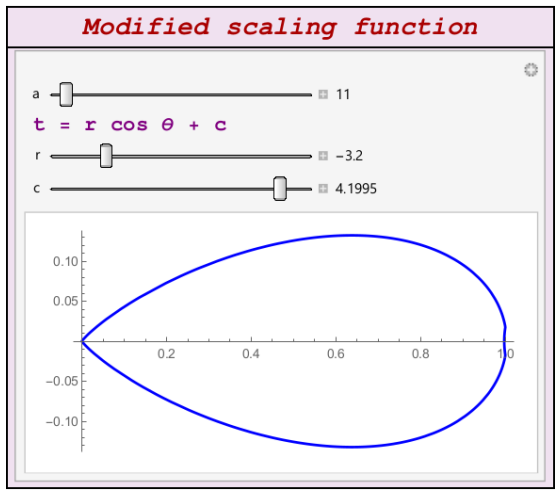


Figure 6: The modified family $\overline{\psi}_a(\theta)$ for $a = 11,\ r = -3.2,\ c = 4.1995$.

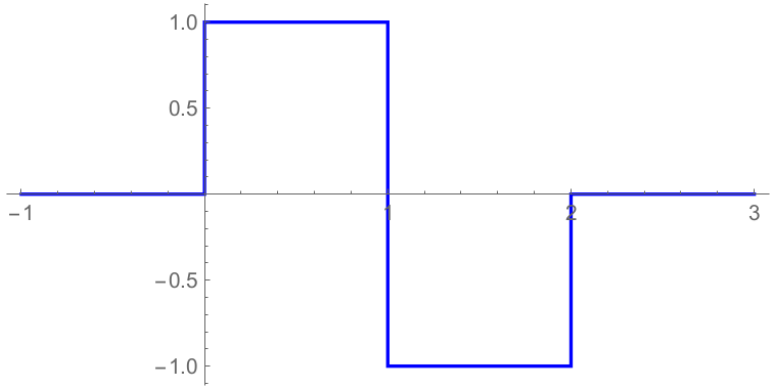


Figure 7: A typical impulse signal from antenna–feeder technique $D(t)$.

2.3. DIFFERENTIAL SCALING FUNCTION. APPLICATIONS

Consider a typical impulse signal from antenna–feeder technique – $D(t)$ (Figure 7).

Define the following *"differential adaptive function"*:

$$\tilde{\psi}_a(t) = \frac{1}{1+e^{-at}} - 2\frac{1}{1+e^{-a(t-1)}} + \frac{1}{1+e^{-a(t-2)}}. \tag{10}$$

In some cases the "differential adaptive function" $\tilde{\psi}_a(t)$ can be used for analysis of signal $D(t)$.

Following the ideas given in Theorem 7, the reader may formulate the corresponding approximation problem and here will be omitted.

The modified family $\tilde{\psi}_a(t)$, given in (10), for two cases: $a = 30$ and $a = 50$ is depicted on Figure 8. Also we give the corresponding values of Hausdorff distance d.

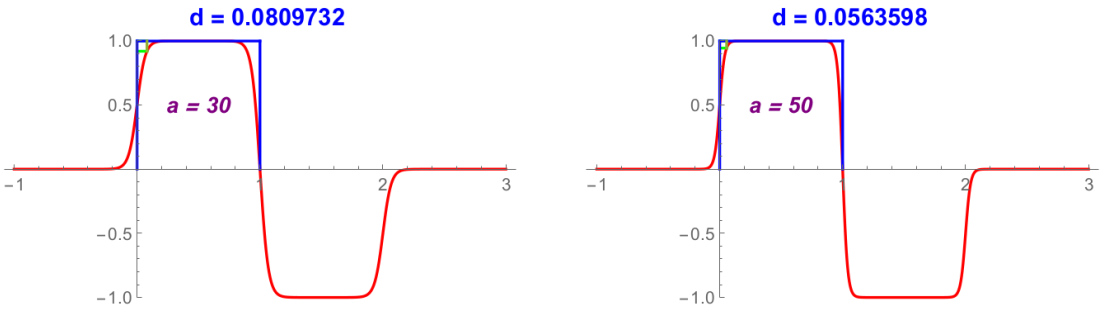


Figure 8: Approximation of the modified family $\tilde{\psi}_a(t)$ by $\psi_a(t)$ for different values of parameter a and corresponding Hausdorff distance d

Let $t = r\cos\theta + c$, $r = 2.4$, $c = -0.1$ and $a = 20$ a typical pattern using $\tilde{\psi}_a(\theta)$ is presented on Figure 9.

Some of the defined *"adaptive function"* can be very useful for approximation of specific data from specialists that are working in scientific fields like Biostatistics, Population Dynamics and Debugging Theory. For other results see related articles and monographs [3, 4, 5, 11, 13, 14, 15, 18, 20, 21, 22, 24, 25, 31, 32, 33, 34, 37, 38, 10, 12, 16, 17, 19, 23, 26, 27, 28, 29, 30, 35, 36] and references there in.

3. CONCLUDING REMARKS

The main task in this work is connected to the approximation of the Haar scaling function by sigmoidal scaling function about Hausdorff metric. Besides, we consider a modified scaling function and differential scaling function with their applications. We prove upper and lower estimates for the searching Hausdorff approximation. For the purpose of this work, a simple dynamic software module is developed and some numerical examples are presented. We use programming environment *CAS Wolfram Mathematica* and *Wolfram Cloud Open Access* to implement obtained results.

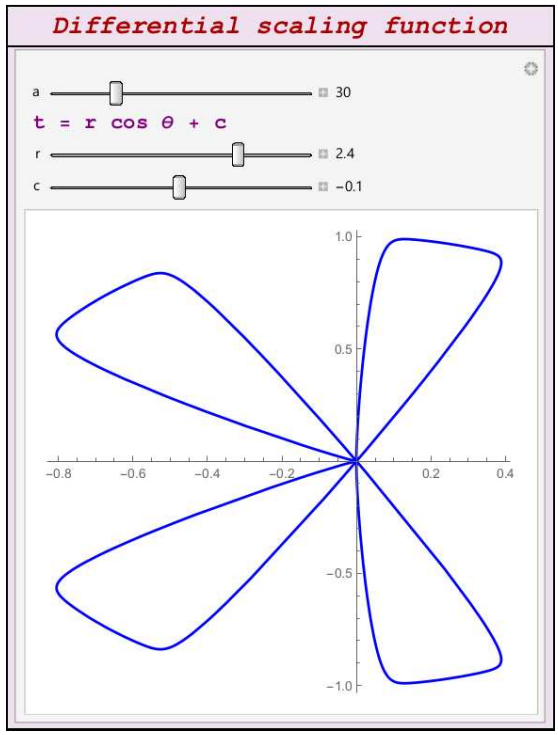


Figure 9: The modified family $\tilde{\psi}_a(\theta)$ for $a = 30,\ r = 2.4,\ c = -0.1$.

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