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A LOOK AT SOME TRIGONOMETRIC-G FAMILIES WITH BASELINE INVERTED EXPONENTIAL (CDF). APPLICATIONS

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ABSTRACT: In this article we study some general classes of trigonometric cumulative distribution functions with baseline inverted exponential (cdf). We consider also modified families of "adaptive functions" with "polynomial variable transfer" with applications to the Antenna–feeder Analysis. We study the "saturation" - d in the Hausdorff sense for some special cases of the families. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Numerical examples using *CAS Mathematica*, illustrating our results are given.

AMS Subject Classification: 41A46

Key Words: Sin–G, Tan–G generalized "adaptive function", Hausdorff distance, upper and lower bounds, modified families of functions with "polynomial variable transfer", radiation diagrams and filter characteristics

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1. INTRODUCTION

In the last few years, there have been serious studies in the literature related to the proposed general classes of trigonometric distributions [2]–[13].

Questions related to the synthesis and analysis of transfer functions, radiation diagrams and filters characteristics are elaborated in detail in [14]–[15].

In this article we study the Hausdorff approximation [16] of the Heaviside function $h_{t_0}(t)$ by special trigonometric –G families with baseline inverted exponential cumulative distribution function (cdf).

Particular attention is paid to the possibility of generating generalized adaptive functions with the so-called "polynomial transfer" and the possibilities that are found for simulating radiation diagrams and filter characteristics.

These results will be of interest for specialists in this modern scientific branch.

We will note that the "sine potential correction" can be used to construct other families of adaptive functions for modeling processes in the field of Debugging and Test Theory and Computer Viruses Propagation.

For some modelling and approximation problems, see [17]–[35].

Definition [16]. The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A-t_B|,|x_A-x_B|)$.

Definition. The Heaviside step–function is defined by:

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0, \end{cases}$$

2. MAIN RESULTS

2.1. A LOOK AT SIN–G FAMILY WITH $G(T) = E^{-\frac{B}{T}}$

Consider the following class of SIN–G family

$$T_1(t) = \sin\left(\frac{\pi}{2}e^{-\frac{b}{t}})\right). \tag{1}$$

for $t > 0,\ 0 < b < 1$.

2.1.1. HAUSDORFF APPROXIMATION OF THE HEAVISIDE STEP FUNCTION BY $T_1(T)$

Let t_0 is the solution of the nonlinear equation

$$T_1(t_0) - \frac{1}{2} = 0.$$

Evidently, for the "median level" we have

$$t_0 = -\frac{b}{\ln\left(\frac{2}{\pi}\arcsin(0.5)\right)}.$$

For the "saturation" - d in the Hausdorff sense we find

$$F(d) := T_1(t_0 + d) - 1 + d = 0, \tag{2}$$

We examine the following approximation of $F(d)$ as we use the function

$$\begin{aligned} H(d) &:= -\tfrac{1}{2} + \left(1 + \tfrac{\sqrt{3}}{2b}\arcsin(0.5)\ln^2\left(\tfrac{2}{\pi}\arcsin(0.5)\right)\right)d \\ &\approx -\tfrac{1}{2} + \left(1 + \tfrac{0.547291}{b}\right)d. \end{aligned} \tag{3}$$

Indeed from Taylor expansion, we get $F(d) - H(d) = \mathcal{O}(d^2)$.

The functions $H(d)$ and $F(d)$ are increasing.

This means that $H(d)$ approximates $F(d)$ with $d \to 0$ as $\mathcal{O}(d^2)$ (see Figure 1).

Let $B = 2.1(1 + \frac{0.0547291}{b})$ and

$$d_l = \frac{1}{B};\ \ d_r = \frac{\ln B}{B}$$

then we see that for $b < 1$

$$H(d_l) < 0; \quad H(d_r) > 0.$$

Thus, we prove the following theorem for upper and lower estimates for the Hausdorff approximation d

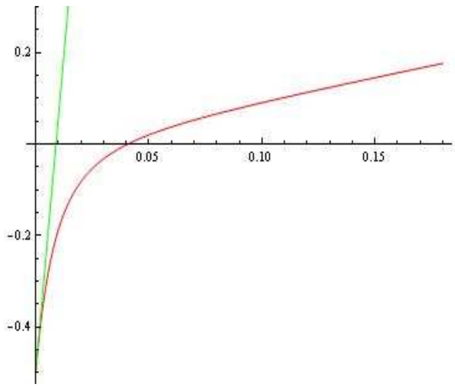

Figure 1: The functions F and H for $b = 0.01$.

Theorem 1. *For the Hausdorff distance d between shifted Heaviside function $h_{t_0}(t)$ and the function $T_1(t)$ the following inequalities hold true:*

$$d_l := \frac{1}{B} < d < \frac{\ln B}{B} := d_r. \tag{4}$$

2.1.2. NUMERICAL EXPERIMENTS

The family $T_1(t)$ for

a) $b = 0.1$; $t_0 = 0.0910239$; Hausdorff distance $d = 0.144276$ (d is computed from nonlinear equation (2)); $d_l = 0.0735667$; $d_r = 0.191977$ (d_l and d_r are computed from inequalities (4));

b) $b = 0.01$; $t_0 = 0.00910239$; Hausdorff distance $d = 0.0406397$; $d_l = 0.00854474$; $d_r = 0.04069$

is plotted on Figure 2.

The investigation of the characteristic "saturation" - d is important.

2.1.3. A FAMILY OF RECURRENCE GENERATED FUNCTIONS BASED ON (1)

We construct a family of recurrence generated functions by

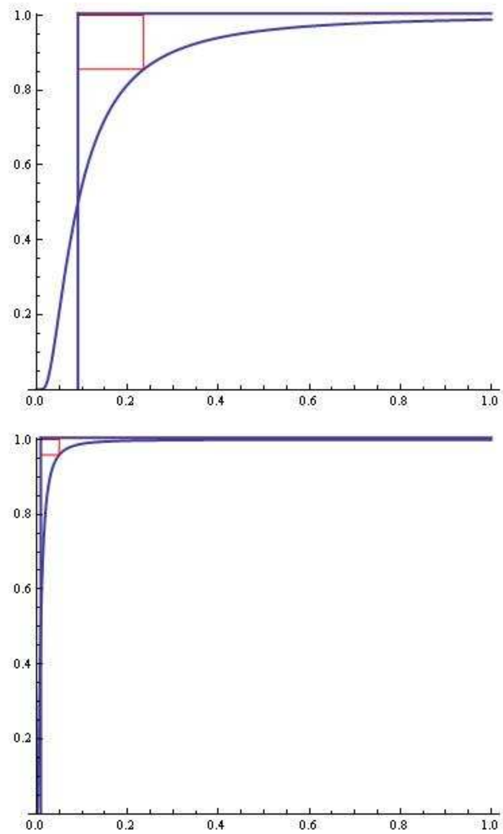

Figure 2: The function T_1 for: a) $b = 0.1$; $t_0 = 0.0910239$; Hausdorff distance $d = 0.144276$; $d_l = 0.0735667$; $d_r = 0.191977$; b) $b = 0.01$; $t_0 = 0.00910239$; Hausdorff distance $d = 0.0406397$; $d_l = 0.00854474$; $d_r = 0.04069$.

$$\phi_{i+1}(t) = \sin\left(\frac{\pi}{2}e^{-\frac{b}{t+\phi_i(t)}}\right) \qquad i = 0, 1, 2, \ldots \tag{5}$$

with

$$\phi_0(t) = T_1(t);\ \ \phi_0(0) = 0.$$

The recurrence generated: $\phi_0(t), \phi_1(t), \phi_2(t), \phi_3(t)$ and $\phi_4(t)$ from family (5) for fixed $b = 0.3$ are visualized on Figure 3.

The new family (5) may find application in the field of Neural Networks.

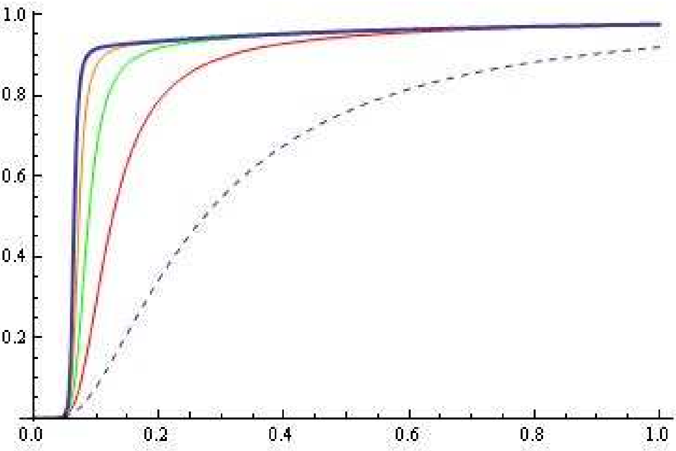

Figure 3: The graphics of recurrence generated adaptive functions: ϕ_0 (dashed), ϕ_1 (red), ϕ_2 (green), ϕ_3 (orange) and ϕ_4 (thick) for fixed $b = 0.3$.

2.1.4. A NEW CLASS OF "ADAPTIVE FUNCTION" WITH "POLYNOMIAL VARIABLE TRANSFER"

Formally, we define the following "adaptive function" with "polynomial variable transfer":

$$T_1^*(t) = \sin\left(\frac{\pi}{2} e^{-\frac{b}{|f(t)|}}\right), \qquad f(t) = \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0. \tag{6}$$

Example. Let $t = r\cos\theta + c$.

Then, typical diagram "Four Leaf Clover" using $T_1^*(\theta)$ for

$n = 5,\ b = 0.3,\ a_0 = 0,\ a_1 = -1.8,\ a_2 = -0.3,\ a_3 = 3.7,\ a_4 = -0.05,\ a_5 = -1,\ r = -0.86,\ c = -1.3$

is plotted on Figure 4.

2.2. A LOOK AT TAN–G FAMILY WITH $G(T) = E^{-\frac{B}{T}}$

Consider the following class of TAN–G family

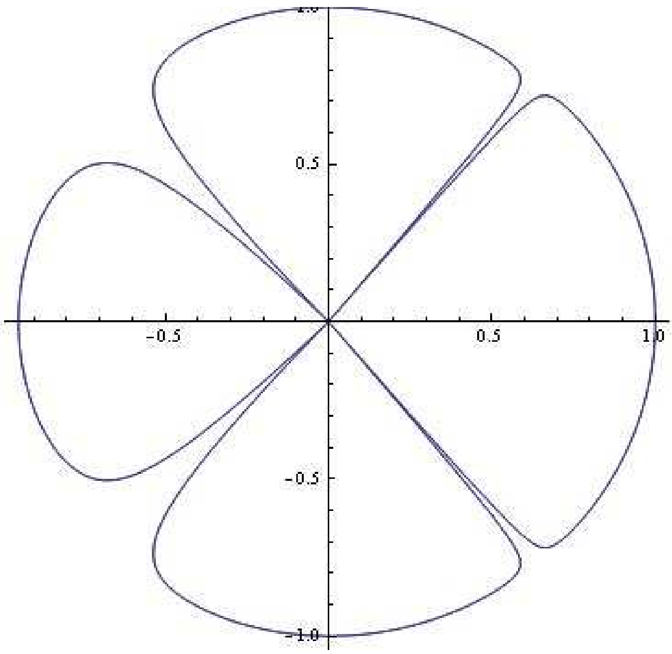

Figure 4: A typical diagram "Four Leaf Clover" using $T_1^*(\theta)$ for $n = 5,\ b = 0.3,\ a_0 = 0,\ a_1 = -1.8,\ a_2 = -0.3,\ a_3 = 3.7,\ a_4 = -0.05,\ a_5 = -1,\ r = -0.86,\ c = -1.3$ in $(-2\pi, 2\pi)$).

$$T_2(t) = \tan\left(\frac{\pi}{4}e^{-\frac{b}{t}})\right). \tag{7}$$

for $t > 0,\ b > 0$.

2.2.1. HAUSDORFF APPROXIMATION OF THE HEAVISIDE STEP FUNCTION BY $T_2(T)$

Let t_0 is the solution of the nonlinear equation

$$T_2(t_0) - \frac{1}{2} = 0.$$

Evidently, for the "median level" we have

$$t_0 = -\frac{b}{\ln\left(\frac{4}{\pi}\arctan(0.5)\right)}.$$

For the "saturation" - d in the Hausdorff sense we find

$$F(d) := T_2(t_0 + d) - 1 + d = 0. \tag{8}$$

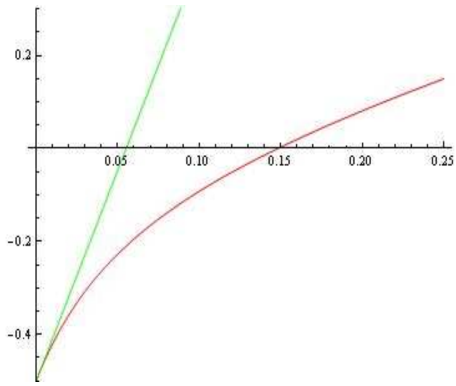

Figure 5: The functions F and H for $b = 0.02$.

We examine the following approximation of $F(d)$ as we use the function

$$H(d) := -\tfrac{1}{2} + \left(1 + \tfrac{5}{4b}\arctan(0.5)\ln^2\left(\tfrac{4}{\pi}\arctan(0.5)\right)\right)d$$
$$\approx -\tfrac{1}{2} + \left(1 + \tfrac{0.161001}{b}\right)d. \quad (9)$$

Indeed from Taylor expansion, we get $F(d) - H(d) = \mathcal{O}(d^2)$.
The functions $H(d)$ and $F(d)$ are increasing.
This means that $H(d)$ approximates $F(d)$ with $d \to 0$ as $\mathcal{O}(d^2)$ (see Figure 5).
Let $C = 2.1(1 + \frac{0.161001}{b})$ and

$$d_l = \frac{1}{C};\;\; d_r = \frac{\ln C}{C}$$

then we see that for $b > 0.02$

$$H(d_l) < 0; \quad H(d_r) > 0.$$

Thus, we prove the following theorem for upper and lower estimates for the Hausdorff approximation d

Theorem 2. *For the Hausdorff distance d between shifted Heaviside function $h_{t_0}(t)$ and the function $T_2(t)$ the following inequalities hold true:*

$$d_l := \frac{1}{C} < d < \frac{\ln C}{C} := d_r. \quad (10)$$

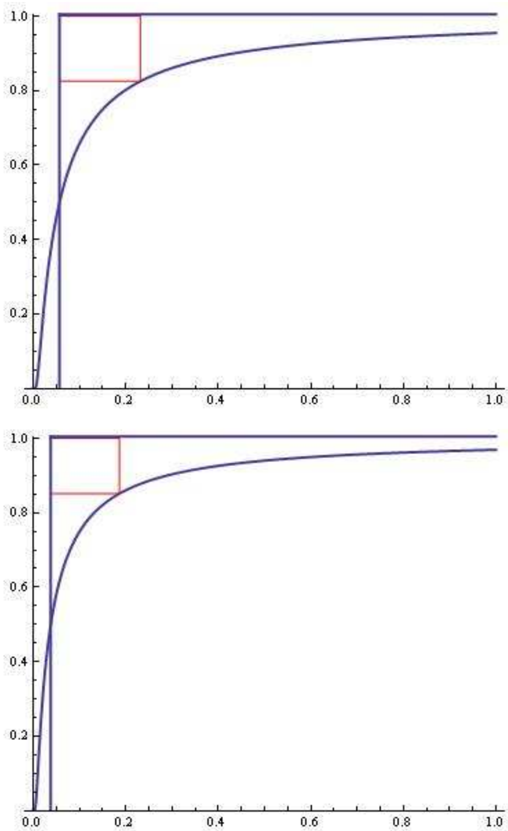

Figure 6: The function T_2 for: a) $b = 0.03$; $t_0 = 0.0569189$; Hausdorff distance $d = 0.174695$; $d_l = 0.0747939$; $d_r = 0.193942$; b) $b = 0.02$; $t_0 = 0.0379459$; Hausdorff distance $d = 0.148352$; $d_l = 0.0526174$; $d_r = 0.154943$.

2.2.2. NUMERICAL EXPERIMENTS

The family $T_2(t)$ for

a) $b = 0.03$; $t_0 = 0.0569189$; Hausdorff distance $d = 0.174695$; $d_l = 0.0747939$; $d_r = 0.193942$;

b) $b = 0.02$; $t_0 = 0.0379459$; Hausdorff distance $d = 0.148352$; $d_l = 0.0526174$; $d_r = 0.154943$

is plotted on Figure 6.

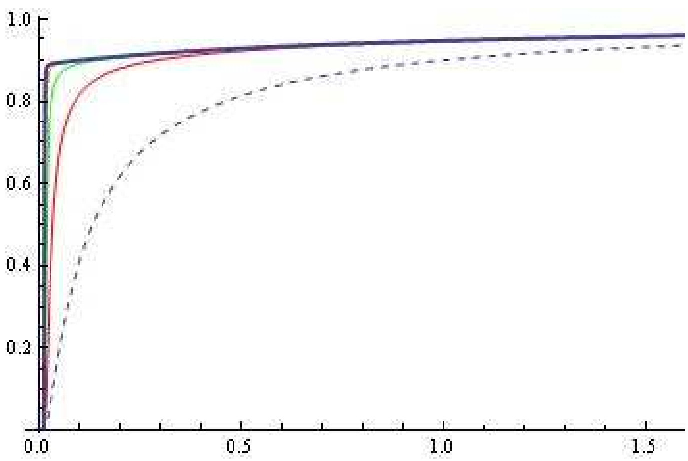

Figure 7: The graphics of recurrence generated adaptive functions: ξ_0 (dashed), ξ_1 (red), ξ_2 (green), ξ_3 (orange) and ξ_4 (thick) for fixed $b = 0.07$.

2.2.3. A FAMILY OF RECURRENCE GENERATED FUNCTIONS BASED ON (7)

We construct a family of recurrence generated functions by

$$\xi_{i+1}(t) = \tan\left(\frac{\pi}{4} e^{-\frac{b}{t+\xi_i(t)}}\right) \quad i = 0, 1, 2, \ldots \tag{11}$$

with

$$\xi_0(t) = T_2(t);\ \ \xi_0(0) = 0.$$

The recurrence generated: $\xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t)$ and $\xi_4(t)$ from family (4.11) for fixed $b = 0.07$ are visualized on Figure 7.

2.2.4. A NEW CLASS OF "ADAPTIVE FUNCTION" WITH "POLYNOMIAL VARIABLE TRANSFER"

Formally, we define the following "adaptive function" with "polynomial variable transfer":

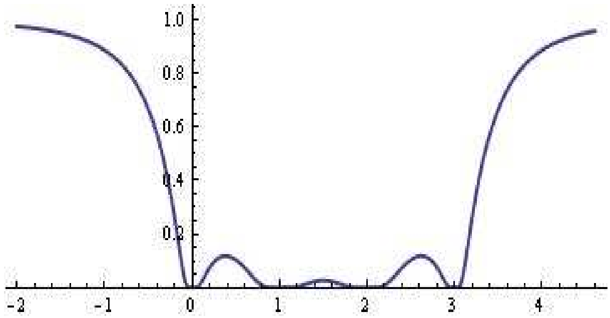

Figure 8: The adaptive function $T_2^*(t)$ for $b = 1.9$ (A typical filter characteristic).

$$\begin{aligned} T_2^*(t) &= \tan\left(\frac{\pi}{4} e^{-\frac{b}{|f(t)|}}\right), \\ f(t) &= \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0. \end{aligned} \tag{12}$$

2.2.5. NUMERICAL SIMULATIONS

Example. Consider the adaptive function

$$T_2^*(t) = \tan\left(\frac{\pi}{4} e^{-\frac{b}{|t(1-t)(2-t)(3-t)|}}\right)$$

For $b = 1.9$ the function $T_2^*(t)$ is depicted on Figure 8.

2.3. A LOOK AT THE FAMILY BY MAHMOOD AND CHESNEAU [8] WITH $G(T) = E^{-\frac{B}{T}}$

Consider the following class of SIN–G family for $t > 0$ and $b > 0$

$$F_1(t) = \sin\left(\frac{\pi}{4} e^{-\frac{b}{t}} (1 + e^{-\frac{b}{t}})\right). \tag{13}$$

The new family for $b := 0.1;\ 0.05;\ 0.001$ is visualized on Figure 9.

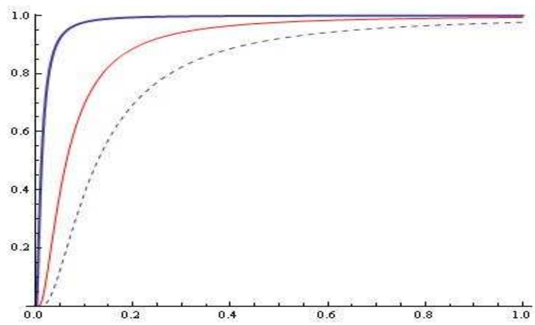

Figure 9: The new family for $b := 0.1;\ 0.05;\ 0.001$.

2.3.1. A NEW CLASS OF "ADAPTIVE FUNCTION" WITH "POLYNOMIAL VARIABLE TRANSFER"

Formally, we define the following "adaptive function" with "polynomial variable transfer":

$$\begin{aligned} F_1^*(t) &= \sin\left(\tfrac{\pi}{4}e^{-\frac{b}{|f(t)|}}(1+e^{-\frac{b}{|f(t)|}})\right), \\ f(t) &= \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0. \end{aligned} \tag{14}$$

Example 1. Let $t = r\cos\theta + c$.

Then, typical diagram using $F_1^*(\theta)$ for

$n = 5,\ b = 0.02,\ a_0 = 0,\ a_1 = -0.1,\ a_2 = 0.2,\ a_3 = 1.7,\ a_4 = -3,\ a_5 = 2.9,\ r = 0.65,\ c = -0.19;$

is plotted on Figure 10.

3. CONCLUDING REMARKS

Using the most popular "power transformation" initiated by Shaw and Buckley [1] we define the following families:

$$M_3(t) = (1+\lambda)\sin\left(\frac{\pi}{2}e^{-\frac{b}{t}}\right) - \lambda\sin^2\left(\frac{\pi}{2}e^{-\frac{b}{t}}\right) \tag{15}$$

and

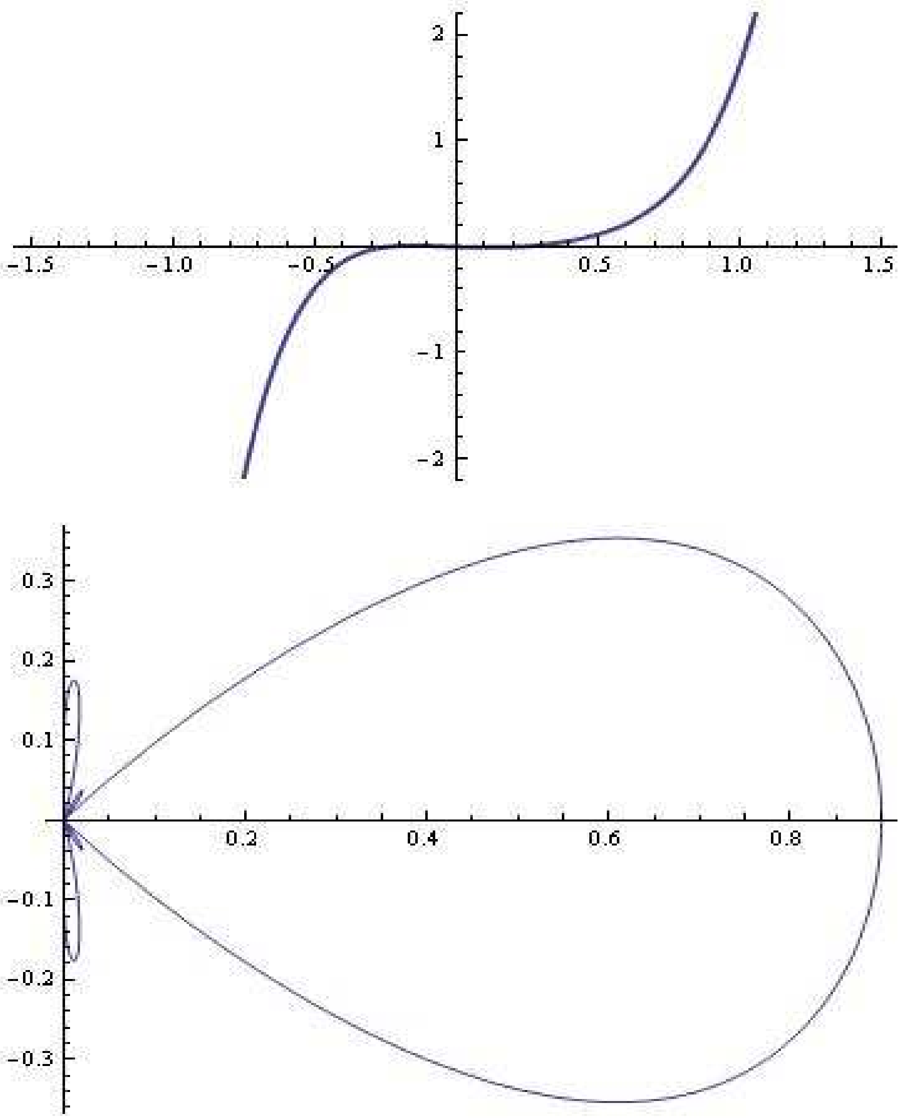

Figure 10: a) The simulation–correction function $f(t)$; b) The family $F_1^*(\theta)$. Example 1.

$$M_4(t) = (1+\lambda)\tan\left(\frac{\pi}{4}e^{-\frac{b}{t}}\right) - \lambda\tan^2\left(\frac{\pi}{4}e^{-\frac{b}{t}}\right) \tag{16}$$

where $|\lambda| < 1$.

The reader may formulate the corresponding approximation problem - investigation on the "saturation" in the Hausdorff sense by using the new models $M_3(t)$ and $M_4(t)$

following the ideas given in this article.

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