International Journal of Differential Equations and Applications
Volume **20**, No. 2 (2021), pages: 207-216
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

RICCI-LIKE SOLITONS ON ALMOST CONTACT B-METRIC MANIFOLDS: A REVIEW

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ABSTRACT: An overview of five papers by the author from the last two years on the topic has been made. Ricci-like solitons are introduced and studied on almost contact B-metric manifolds (also known as almost contact complex Riemannian manifolds) in the cases when the soliton's potential is the Reeb vector field, vertical or arbitrary vector field. The cases of Sasaki-like manifolds and torse-forming potentials have been considered. In these cases, necessary and sufficient conditions are proved these manifolds are (almost) Einstein-like. Explicit examples of Lie groups as 3- and 5-dimensional manifolds with the structures studied are provided. Some generalizations of these solitons are considered: almost Ricci-like solitons and gradient almost Ricci-like solitons.

AMS Subject Classification: 53C25, 53D15, 53C50; 53C44, 53D35, 70G45
Key Words: Ricci-like soliton, η-Ricci soliton, Einstein-like manifold, η-Einstein manifold, almost contact B-metric manifold, almost contact complex Riemannian manifold, torse-forming vector field

Received: August 23, 2021 **Revised:** October 29, 2021
Published: November 15, 2021 **doi:** 10.12732/ijdea.v20i2.7
Academic Publications, Ltd. https://acadpubl.eu

1. INTRODUCTION

In recent years, there has been a growing interest among scientists of differential geometry and theoretical physics on different types of solitons, and especially of Ricci solitons, on manifolds equipped with different additional tensor structures and metrics.

Almost contact B-metric manifolds have been introduced and studied in the last 30 years mainly by Bulgarian geometers. For this reason, it is natural for the author to be interested in introducing geometric objects of type of Ricci solitons on these manifolds.

The main difference of Ricci-like solitons on the studied manifolds in comparison with Ricci solitons on other manifolds is the presence of another metric associated with the basic one through the structure of the manifold.

In this way, the type of solitons studied generalize the usual Ricci solitons and η-Ricci solitons.

2. ALMOST CONTACT B-METRIC MANIFOLDS

We consider an *almost contact B-metric manifold* $(M, \varphi, \xi, \eta, g)$, $\dim M = 2n+1$. The almost contact structure is well-known. The B-metric g has the property $g(\varphi\cdot, \varphi\cdot) = -g + \eta\otimes\eta$, and together with its associated B-metric $\tilde{g}$, defined by $\tilde{g} = g(\cdot, \varphi\cdot) + \eta\otimes\eta$, have a signature $(n+1, n)$. A classification of these manifolds of eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\dots$, $\mathcal{F}_{11}$ is given in [2].

2.1. SASAKI-LIKE ALMOST CONTACT B-METRIC MANIFOLDS

In [4], by the condition its complex cone to be a Kähler-Norden manifold, the class of *Sasaki-like spaces* it is defined in the set of almost contact B-metric manifolds, also known as almost contact complex Riemannian (abbreviated accR) manifolds. Then, these Sasaki-like spaces are determined by the condition $(\nabla_x\varphi)\, y = g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x$.

Obviously, they form a subclass of $\mathcal{F}_4$ determined by Lee forms of the form $\theta = -2n\,\eta$, $\theta^* = \omega = 0$.

For an explicit example of a Sasaki-like manifold see Example 2 in [4].

2.2. EINSTEIN-LIKE ALMOST CONTACT B-METRIC MANIFOLDS

In [6], it is introduced the notion of an *Einstein-like* manifold if its Ricci tensor ρ satisfies the property $\rho = a\,g + b\,\tilde{g} + c\,\eta\otimes\eta$ for some triplet of constants (a, b, c).

Consequently, the corresponding scalar curvatures of g and $\tilde{g}$ are $\tau = (2n+1)a+b+c$ and $\tilde{\tau} = 2n(b+1)$, respectively.

If a, b, c are functions on M, then M is called *almost Einstein-like*, *almost η-Einstein* and *almost Einstein*, respectively.

Particularly, when $b = 0$ and $b = c = 0$, the manifold is called an *η-Einstein manifold* and an *Einstein manifold*, respectively. Other special cases of Einstein-like types of the considered manifolds are given in [11] and [5].

For an Einstein-like manifold, there are found the properties of ρ with respect to the structure and the expression of $\nabla\rho$.

Proposition 1 ([6]). For a Sasaki-like Einstein-like (M,φ,ξ,η,g) we have:

$$a+b+c=2n, \qquad \tau=2n(a+1),$$

$$\begin{aligned}(\nabla_x\rho)(y,z) = &-(b+c)\left\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\right\}\\ &+b\left\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\right\}.\end{aligned}$$

$$\begin{aligned}a&=2n+\frac{1}{2n}(\operatorname{div}^*\rho)(\xi), \qquad b=-\frac{1}{2n}(\operatorname{div}\rho)(\xi),\\ c&=\frac{1}{2n}\left\{(\operatorname{div}\rho)(\xi)-(\operatorname{div}^*\rho)(\xi)\right\},\end{aligned}$$

where div and div* denote the divergence with respect to g and $\tilde{g}$, respectively.

Corollary 2 ([6]). *For a Sasaki-like Einstein-like (M,φ,ξ,η,g) we have:*

(i) It is scalar-flat if and only if $a=-1$.

(ii) It is Ricci-symmetric if and only if it is an Einstein manifold.

(iii) Its Ricci tensor is η-parallel and parallel along ξ.

(iv) It is η-Einstein if and only if $\operatorname{div} Q \in \ker\eta$ *for the Ricci operator Q.*

(v) It is Einstein if and only if $\operatorname{div} Q, \operatorname{div}^* Q \in \ker\eta$.

Example of an Einstein-like manifold is the example in § 2.1, for which we find that $\rho = 4\eta\otimes\eta$, i.e. the constructed manifold is η-Einstein.

Proposition 3 ([8]). If (M,φ,ξ,η,g) is Sasaki-like and almost Einstein-like, then its scalar curvatures τ and $\tilde{\tau}$ are constants $\tau = const$, $\tilde{\tau}=2n$ and it is η-Einstein with constants $(a,b,c)=(\tau/2n-1,\ 0,\ 2n+1-\tau/2n)$.

2.3. ACCR MANIFOLDS WITH A TORSE-FORMING REEB VECTOR FIELD

In this case, ξ satisfies the property $\nabla_x\xi = -f\,\varphi^2 x$, where the function f is non-identically zero. Then, such accR manifolds belong to the class $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$. Among the basic classes, only $\mathcal{F}_5$ can contain them. As an example of such an accR manifold, we show those from Theorem 2.1 of [5], where a 3-dimensional $\mathcal{F}_5$-manifold on a Lie group is given with a constant $p = -\frac{1}{2}\theta^*(\xi)$. This manifold is Einstein because $\rho = -2p^2 g$ holds. Moreover, ξ is torse-forming with constant $f = \frac{1}{2}\theta^*(\xi)$.

3. RICCI-LIKE SOLITONS ON ACCR MANIFOLDS

Due to the presence of two associated metrics g and $\tilde{g}$ on M, we introduce a generalization of the Ricci soliton and the η-Ricci soliton in [6]:

Definition 4. An accR manifold $(M, \varphi, \xi, \eta, g)$ is called a *Ricci-like soliton* with potential vector field ξ if its Ricci tensor ρ satisfies the condition

$$\tfrac{1}{2}\mathcal{L}_\xi g + \rho + \lambda\, g + \mu\,\tilde{g} + \nu\,\eta\otimes\eta = 0, \qquad (\lambda, \mu, \nu) \in \mathbb{R}^3. \tag{1}$$

In particular, for $\mu = \nu = 0$ and $\nu = 0$, a *Ricci soliton* and an *η-Ricci soliton* are obtained, defined in [3] and [1], respectively.

Proposition 5. Let $(M, \varphi, \xi, \eta, g)$ with a Ricci tensor ρ be Einstein-like with constants (a, b, c) and let the manifold admit a Ricci-like soliton with potential ξ and constants (λ, μ, ν). Then the following properties are valid:

(i) $a + b + c + \lambda + \mu + \nu = 0$;

(ii) ξ is geodesic, i.e. $\nabla_\xi\xi = 0$;

(iii) $(\nabla_\xi\varphi)\,\xi = 0,\ \nabla_\xi\eta = 0,\ \omega = 0$;

(iv) $(\nabla_\xi\rho)\,(y, z) = b\,g\big((\nabla_\xi\varphi)\,y, z\big)$.

Corollary 6. *Let $(M, \varphi, \xi, \eta, g)$ satisfy the hypothesis of Proposition 5. Then, we have:*

(i) The manifold does not belong to $\mathcal{F}_{11}$ or to its direct sum with other basic classes.

(ii) The Ricci tensor is parallel along ξ if and only if the manifold does not belong to $\mathcal{F}_{10}$ or to its direct sum with other basic classes.

(iii) The Ricci tensor is parallel along ξ if and only if the manifold is η-Einstein.

3.1. RICCI-LIKE SOLITONS ON SASAKI-LIKE ACCR MANIFOLDS

Theorem 7 ([6]). *Let six constants meet the conditions $a+\lambda = b+\mu-1 = c+\nu+1 = 0$ for a Sasaki-like accR manifold M. Then, M admits a Ricci-like soliton with potential ξ and constants (λ,μ,ν), where $\lambda+\mu+\nu = -2n$, if and only if it is Einstein-like with constants (a,b,c), where $a+b+c = 2n$.*

In particular, we get:

(i) M admits an η-Ricci soliton with potential ξ and constants $(\lambda, -2n-\lambda)$ if and only if M is Einstein-like with constants $(-\lambda, 1, \lambda+2n-1)$.

(ii) M admits a shrinking Ricci soliton with potential ξ and constant $-2n$ if and only if M is Einstein-like with constants $(2n, 1, -1)$.

(iii) M is η-Einstein with constants $(a, 2n-a)$ if and only if M admits a Ricci-like soliton with potential ξ and constants $(-a, 1, a-2n-1)$.

(iv) The manifold is Einstein with constant $2n$ if and only if it admits a Ricci-like soliton with potential ξ and constants $(-2n, 1, -1)$.

Corollary 8 ([7]). *Let (M,φ,ξ,η,g) satisfy the conditions in the general case of Theorem 7. Then, the six constants are expressed by scalar curvatures τ and $\tilde{\tau}$ with respect to g and $\tilde{g}$, respectively, as follows*

$$\lambda = 1-\tau/(2n), \quad \mu = 2-\tilde{\tau}/(2n), \quad \nu = (\tau+\tilde{\tau})/(2n)-2n-3,$$
$$a = \tau/(2n)-1, \quad b = \tilde{\tau}/(2n)-1, \quad c = 2n+2-(\tau+\tilde{\tau})/(2n).$$

The example from §2.1. and §2.2 has $(\lambda,\mu,\nu) = (0,1,-5)$ and the constructed manifold admits a Ricci-like soliton with potential ξ and it supports the case (iii) of Theorem 7.

Proposition 9 ([7]). Every Einstein-like Sasaki-like manifold (M,φ,ξ,η,g) admitting a Ricci-like soliton with potential ξ is Ricci η-parallel and Ricci parallel along ξ, i.e. $(\nabla\rho)|_{\ker\eta} = 0$ and $\nabla_\xi\rho = 0$, respectively.

Proposition 10 ([7]). Let (M,φ,ξ,η,g) be a Sasaki-like admitting a Ricci-like soliton with potential ξ. The manifold is locally Ricci symmetric if and only if $(\lambda,\mu,\nu) = (-2n,1,-1)$, i.e. it is an Einstein manifold.

It is clear that a Sasaki-like manifold is locally Ricci symmetric just in the case (iv) of Theorem 7.

Proposition 11. Let (M,φ,ξ,η,g) satisfy the conditions of Proposition 10 for $(\lambda,\mu)\neq(0,1)$. Then its Ricci tensor ρ is ∇-recurrent and satisfies

$$\begin{aligned}(\nabla_x\rho)\,(y,z)=&A\{(1-\mu)^2+\lambda(\lambda+2n)\}\{\rho(x,\varphi y)\eta(z)+\rho(x,\varphi z)\eta(y)\}\\&-2A(\lambda+n)(1-\mu)\{\rho(\varphi x,\varphi y)\eta(z)+\rho(\varphi x,\varphi z)\eta(y)\},\end{aligned}$$

where $A=\{\lambda^2+(1-\mu)^2\}^{-1}$.

3.2. RICCI-LIKE SOLITONS WITH POTENTIAL POINTWISE COLLINEAR WITH ξ

Let (M,φ,ξ,η,g) admit a *Ricci-like soliton with potential* v, i.e. ξ is replaced by v in (1). Suppose that (M,φ,ξ,η,g) is Sasaki-like and v is pointwise collinear with ξ, i.e. $v=k\,\xi$, where k is a differentiable function. The results of this section are given in [7].

Theorem 12. *For the considered manifold (M,φ,ξ,η,g), we have $k=\mu$, i.e. k is constant, the equality $\lambda+\nu=-k-2n$ is valid and M is η-Einstein with constants $(a,b,c)=(-\lambda,0,\lambda+2n)$.*

Theorem 13. *The studied manifold (M,φ,ξ,η,g) has the following additional curvature properties:*

(i) It is locally Ricci symmetric if and only if it is Einstein.

(ii) It is Ricci semi-symmetric, i.e. $R(.\,,.)\cdot\rho=0$, if and only if it is Einstein.

(iii) Its Ricci tensor is cyclic parallel, i.e.

$$(\nabla_x\rho)(y,z)+(\nabla_y\rho)(z,x)+(\nabla_z\rho)(x,y)=0$$

if and only if it is Einstein.

(iv) Its Ricci tensor is of a Codazzi type, i.e.

$$(\nabla_x\rho)(y,z)=(\nabla_y\rho)(x,z),$$

if and only if the manifold is Einstein.

(v) It is locally Ricci φ-symmetric, i.e. $\varphi^2(\nabla_xQ)y=0$ for $x,y\perp\xi$.

(vi) *It is globally Ricci φ-symmetric, i.e. $\varphi^2(\nabla_x Q)y = 0$ for arbitrary x and y, if and only if it is Einstein.*

(vii) *It is almost pseudo Ricci symmetric, i.e.*

$$(\nabla_x \rho)(y, z) = \{\alpha(x) + \beta(x)\}\rho(y, z) + \alpha(y)\rho(x, z) + \alpha(z)\rho(x, z)$$

for non-vanishing 1-forms α and β, if and only if the manifold is Einstein.

(viii) *It is special weakly Ricci symmetric, i.e.*

$$(\nabla_x \rho)(y, z) = 2\alpha(x)\rho(y, z) + \alpha(y)\rho(x, z) + \alpha(z)\rho(x, y)$$

for a non-vanishing 1-form α, if and only if the manifold is Einstein.

(ix) *It does not admit a Ricci-like soliton with vertical potential if it has the curvature property $Q \cdot R = 0$.*

Proposition 14. On an arbitrary Sasaki-like manifold, every symmetric second-order covariant tensor that is parallel with respect to ∇, is a constant multiple of g.

The latter assertion is applied to a Ricci-like soliton in the following

Theorem 15. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasaki-like manifold of dimension $2n + 1$ and let h be the tensor $\frac{1}{2}\mathcal{L}_\xi g + \rho + \mu\,\tilde{g} + \nu\,\eta \otimes \eta$, where $\mu, \nu \in \mathbb{R}$. The tensor h is parallel with respect to ∇ of g if and only if $(M, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential ξ and constants (λ, μ, ν), where $\lambda = -h(\xi, \xi) = -\mu - \nu - 2n$.*

The constructed 5-dimensional example, commented in §2.1, §2.2 and §2.3., supports also Theorem 7, Theorem 12 for $k = 1$ and it agrees with Proposition 14 and Theorem 15 for the trivial case $h = 0$.

3.3. RICCI-LIKE SOLITONS WITH ARBITRARY POTENTIAL

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like manifold. The following results are obtained in [8].

Theorem 16. *If $(M, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with arbitrary potential v then we have the following properties: $\lambda + \mu + \nu = -2n$, $\nabla_\xi v = -\varphi v$, $(\mathcal{L}_v \rho)(x, \xi) = 0$, $\tau = 2n$ and $\tilde{\tau} = const.$ Suppose that $(M, \varphi, \xi, \eta, g)$ is additionally Einstein-like then we have $\rho = 2n\,\eta \otimes \eta$ and $\tau = \tilde{\tau} = 2n$; as a corollary it is η-Einstein with constants $(0, 0, 2n)$, which is equivalent to the existence on M of a Ricci-like ξ-soliton and constants $(0, 1, -2n - 1)$.*

Theorem 17. *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with potential v then the sectional curvatures of its φ-holomorphic sections (resp., ξ-sections) are equal to -1 (resp., 1).*

A 3-dimensional example is given as Example 2 in [8]. It is an Einstein-like manifold with constants $(a, b, c) = (0, 0, 2)$. A potential v is determined explicitly in a general position depending on arbitrary constants c_1, c_2, c_3. The corresponding Ricci-like soliton has the constants $\lambda = c_1$, $\mu = c_2 + c_3$, $\nu = -c_1 - c_2 - c_3 - 2$.

4. RICCI-LIKE SOLITONS WITH TORSE-FORMING POTENTIAL

Theorem 18. *The manifold $(M, \varphi, \xi, \eta, g)$ with torse-forming ξ admits a Ricci-like soliton with potential ξ if and only if M is Einstein-like and*

$$a + \lambda + f = 0, \quad b + \mu = 0, \quad c + \nu - f = 0, \quad f = \text{const}.$$

In particular, we have:

(i) M admits an η-Ricci soliton with potential ξ and constants (λ, ν) if and only if M is η-Einstein with constants (a, c) with $a + c + \lambda + \nu = 0$.

(ii) M admits a Ricci soliton with potential ξ and constant λ if and only if M is η-Einstein with constants $(-\lambda - f, f)$.

(iii) M is Einstein with constant a if and only if M admits an η-Ricci with potential ξ and constants $(-a - f, f)$.

Proposition 19. Let $(M, \varphi, \xi, \eta, g)$ be Einstein-like, which admits a Ricci-like soliton with torse-forming potential ξ. Then f is determined by:

$$2n\, f^2 = -(a + b + c) = \lambda + \mu + \nu.$$

Thus $(M, \varphi, \xi, \eta, g)$ has constant negative ξ-sectional curvatures $-f^2$.

Corollary 20. *The manifold $(M, \varphi, \xi, \eta, g)$ is Einstein with $\tau < 0$ if and only if M admits an η-Ricci soliton with torse-forming potential ξ and constants $(\lambda = -\tau/(2n+1) - f, \nu = f)$ for constant $f = \pm\sqrt{-\tau/(2n(2n+1))}$.*

Example of a Ricci-like soliton with torse-forming ξ is the $\mathcal{F}_5$-manifold from § 2.3, where ξ is torse-forming with constant $f = -p$, and it supports the case (iii) of Theorem 18 and the rest results in § 4.

5. FURTHER RESEARCH ON RICCI-LIKE SOLITONS OF ACCR MANIFOLDS

The investigations on Ricci-like solitons and their generalizations considered on accR manifolds continues with the second part of [8] as well as [9] and [10].

In [8], an almost Ricci-like soliton is considered, i.e. λ, μ, ν are functions on M. If its potential v is a gradient of a differentiable function f, i.e. $v = \operatorname{grad} f$, then the soliton is called a *gradient almost Ricci-like soliton* of $(M, \varphi, \xi, \eta, g)$. It is proved the following

Theorem 21. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasaki-like accR manifold of dimension $2n + 1$. If it admits a gradient almost Ricci-like soliton with functions (λ, μ, ν) and a potential function f, then $(M, \varphi, \xi, \eta, g)$ has constant scalar curvatures $\tau = \tilde{\tau} = 2n$ and its Ricci tensor is $\rho = 2n\, \eta \otimes \eta$.*

For the 3-dimensional example mentioned in §3.3., it is determined f by $f = -\frac{1}{2}s\,\{(x^1)^2 + (x^2)^2\} + x^2 + t\,x^3$ for arbitrary constants s and t. Then we have $v = \operatorname{grad} f = -\,\{s\,x^1 \cos x^3 + (s\,x^2 - 1)\sin x^3\}e_1 + \{s\,x^1 \sin x^3 - (s\,x^2 - 1)\cos x^3\}e_2 + t\,e_3$ and the potential constants are $\lambda = s$, $\mu = t$, $\nu = -s - t - 2$. The obtained gradient almost Ricci-like soliton supports Theorem 21 for $n = 1$.

In [9], we generalize and develop our study of Ricci-like solitons on accR manifolds by studying a potential that is torse-forming with constant length and vertical direction – in the sense that it is orthogonal to the contact (horizontal) distribution $\ker(\eta)$ with respect to the basic metric. The main results are the following. We prove necessary and sufficient conditions for the studied manifolds to admit almost Ricci-like solitons and to be almost Einstein-like manifolds. Then, we give a characterization for almost Ricci-like solitons on the studied manifold concerning a parallel symmetric $(0, 2)$-tensor. Next, we construct an explicit example of a smooth manifold of dimension $(2n+1)$ equipped with an accR structure. Then, we show that the manifold of the considered type and the results for this example support the relevant assertions in the previous sections.

In [10], we study curvature properties of almost Ricci-like solitons whose potential is torse-forming and vertical on accR manifolds. The main results here are a series of properties of the curvature tensor and its Ricci tensor on the manifolds under study such as those in Theorem 13 and an explicit example of a smooth manifold of the considered type with comments on the results for this example supporting the relevant assertions.

ACKNOWLEDGMENTS

This work is supported by the project FP21-FMI-002 of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

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