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A NOTE ON THE EXTENDED GOMPERTZ GROWTH MODELS

Nikolay Pavlov[1], Vesselin Kyurkchiev[1],
Anton Iliev[1,2], Asen Rahnev[1], and Nikolay Kyurkchiev[1,2]

[1]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

[2]Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, BULGARIA

ABSTRACT: Following the ideas given in [18], in this article we study a hypothetical piecewise smooth extended Gompertz growth function $B(b_1(t), b_2(t))$. More precise we study the saturation with the new class to the horizontal asymptote with respect to the Hausdorff distance. Some numerical examples, using *CAS MATHEMATICA* are also given. Studies in this paper can also be applied to a random shifted sigmoidal functions of generalized Gompertz–G families.

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Key Words: hypothetical piecewise smooth extended Gompertz growth function, Hausdorff distance, lower bound

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1. INTRODUCTION

Gompertz distribution [1] plays an important role in many real life applications such as reliability, biostatistics, population dynamics, impulsive analysis, human mortality and actuarial sciences.

Gompertz model makes an extensive use of the sigmoidal function:

$$G(t) = Ae^{-e^{-kt}}. \tag{1}$$

There are many and varied attempts to summarize and modify it in view of its specific application in various branches of science. For more details, see [2]–[15], [24]–[25].

In [15] the author consider the following extended sigmoidal growth function based on Gompertz

$$g(t) = \left(1 - e^{-kt}\right) e^{-e^{-kt}}, \quad t > 0,\ k > 0. \tag{2}$$

In this paper we will discuss some features of this new family.

Definition. [16]. The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f, g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A - B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

More precise we study the saturation with the new class (2) to the horizontal asymptote with respect to the Hausdorff distance d [16].

The hypothetical piecewise smooth sigmoidal growth model associated to the model (2) is also given.

2. MAIN RESULTS.

2.1. INVESTIGATIONS ON THE MODEL (2).

The following is valid

Theorem. For sufficiently large values of $k \geq 30$, for the "saturation" - d we have

$$2d^* := 2\frac{-e - k + \sqrt{2ek^2 + (e + k)^2}}{k^2} < d. \tag{3}$$

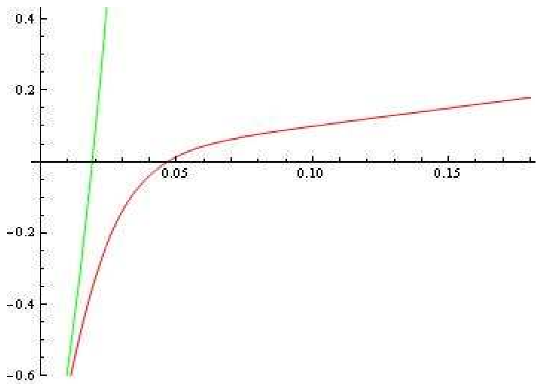

Figure 1: The functions $F(d)$ -red and $H(d)$ -green for fixed $k = 80$.

Sketch of the proof.

For the saturation d we have

$$F(d) := \left(1 - e^{-kd}\right) e^{-e^{-kd}} - 1 + d = 0. \tag{4}$$

Insofar as the proof is based on a technique proposed in [17], we will only note that the function:

$$H(d) := -1 + \left(1 + \frac{k}{e}\right) d + \frac{k^2 d^2}{2e} \tag{5}$$

approximates $F(d)$ with $d \to 0$ as $\mathcal{O}(d^3)$ (see, for example Figure 1).

From the Descartes rule of signs we find that the equation $H(d) = 0$ has unique positive root:

$$d^* := \frac{-e - k + \sqrt{2ek^2 + (e + k)^2}}{k^2}.$$

After a precise analysis for $k \geq 30$ we get the estimate (3).

For example, for fixed $k = 30$ we find $d = 0.0989344$ and for $k = 100$ we have $d = 0.0391788$ (see, Figure 2).

Some computational examples using (3) and (4) are presented in Table 1.

k	$2d^*$	d *computed by* (4)
30	0.0988994	0.0989344
50	0.0601836	0.0673139
80	0.0379161	0.046733
100	0.0304138	0.0391788
150	0.0203482	0.0283125
250	0.0122438	0.0186682

Table 1: Bounds for d for various values of k.

2.2. THE HYPOTHETICAL PIECEWISE SMOOTH SIGMOIDAL GROWTH FUNCTION ASSOCIATED TO THE MODEL (2).

Consider the following modification of the extended Gompertz model (2):

$$b_1(t) = (A - e^{-kt})e^{-e^{-kt}}. \tag{6}$$

Without community restriction, we will consider model (6) in the constraint: $b_1(0) = (A-1)e^{-1} = \frac{1}{2}$, i.e. $A = 1 + \frac{e}{2}$.

For a visualization of this model at fixed value of the parameter k, see Figure 3.

Suppose for a moment that the researcher wants a fixed model $b_1(t)$ and parameters A and k to achieve a lower level of saturation $B = (A - e^{-1})e^{-e^{-1}}$ when approximating the experimental data.

This can be achieved, for example, with the function $b_2(t)$ (for $t > 0$, see Figure 4)

$$b_2(t) = (A - e^{-\frac{kt}{1+kt}})e^{-e^{-\frac{kt}{1+kt}}}. \tag{7}$$

To construct the function $b_2(t)$, we again used our familiar "fractional linear correction".

Definition. This leads us to think of the following hypothetical piecewise smooth sigmoidal growth function

$$B(t) := \begin{cases} (A - e^{-kt})e^{-e^{-kt}} := b_1(t), & t < 0 \\ \frac{1}{2}, & t = 0 \\ (A - e^{-\frac{kt}{1+kt}})e^{-e^{-\frac{kt}{1+kt}}} := b_2(t), & t > 0. \end{cases} \tag{8}$$

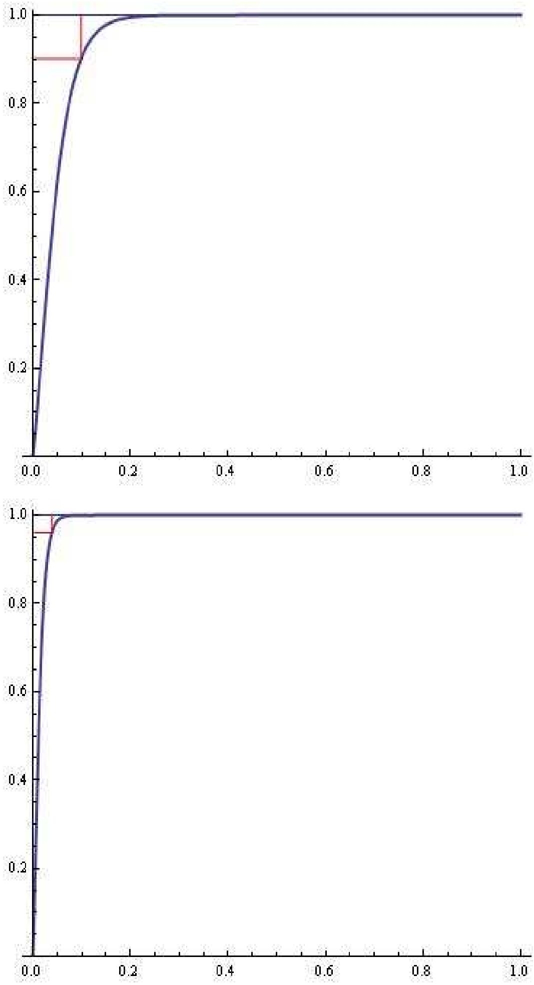

Figure 2: The "saturation" - d for a) $k = 30, d = 0.0989344$; b) $k = 100, d = 0.0391788$.

Evidently, from (8) we have

$$b_1'(0) = b_2'(0).$$

The function $B(b_1(t), b_2(t))$ for fixed $k = 10$, $A = 2.35514$, $B = 1.37835$ is depicted on Fig. 5c.

2.3. CONCLUDING REMARKS

1. We will explicitly note that the estimate (3) may be useful for users due to the

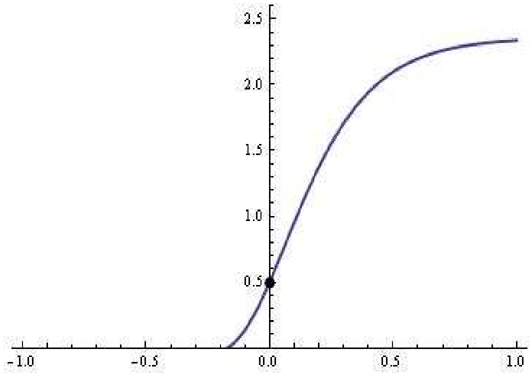

Figure 3: The function $b_1(t)$ for fixed $k = 10$, $b_1(0) = \frac{1}{2}$.

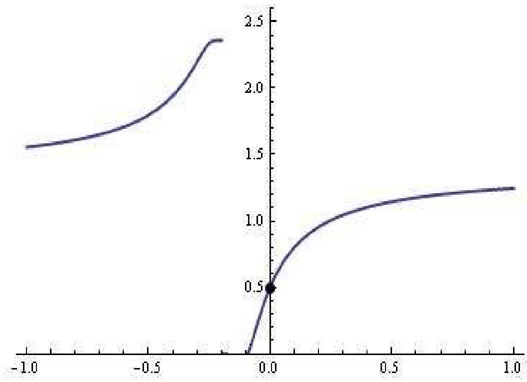

Figure 4: The function $b_2(t)$ for fixed $k = 10$, $(b_2(0) = \frac{1}{2})$.

fact that the adaptation of this model in an arbitrary Computer Algebraic System presupposes the knowledge of an appropriate initial approximation for the root of the nonlinear equation (4), and, moreover, it is necessary double precision operation.

2. Studies in this paper can also be applied to a random shifted sigmoidal functions of generalized Gompertz–G families, see for example the family [25]:

$$gg(t) = \left(1 - e^{-(kt)^{k_1}}\right) e^{-e^{-(kt)^{k_1}}}, \quad t > 0,\ k > 0,\ k_1 > 0. \tag{9}$$

In some cases, saturation with the model (9) is faster, see Figure 6.

3. With the methodological aspects discussed here, we aim to provide the researcher (who does not have to be a mathematician) with a reliable software tool for making adjustments to his chosen model - for the specific experiment.

4. In [19] we develop some dynamic programming modules implemented within

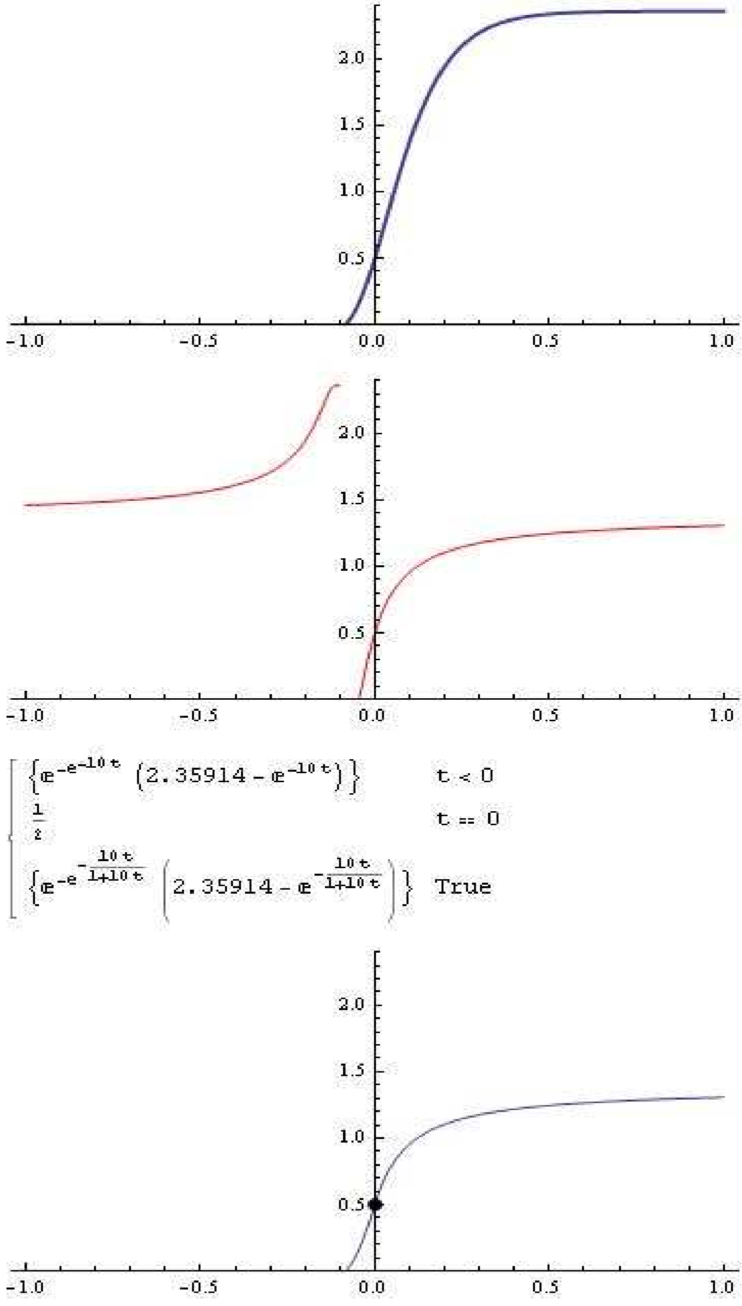

Figure 5: The functions: $b_1(t)$-thick; $b_2(t)$-red and $B(b_1(t), b_2(t))$ for fixed $k = 10$, $A = 2.35514$, $B = 1.37835$.

the programming environment CAS Wolfram Mathematica and Wolfram Cloud Open Access.

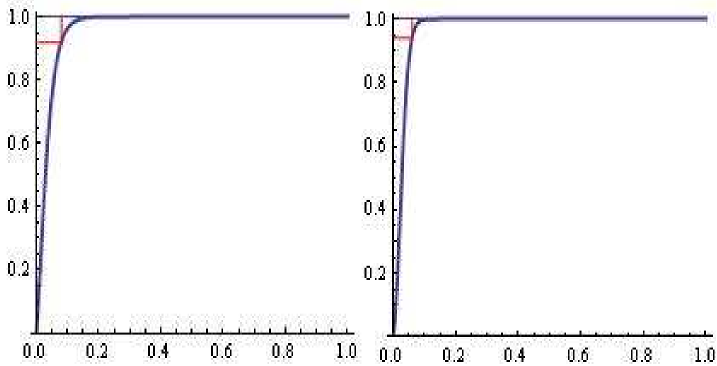

Figure 6: Comparisons between (2) and (9): a) For the model (2): The ”saturation” for $k = 40$ is $d = 0.0797737$; b) For the model (9): The ”saturation” for $k = 40$, $k_1 = 1.4$ is $d_1 = 0.0607987$.

It is planned to upgrade the *Distributed Platform for e-Learning - DisPeL* in view of the possibility of covering emerging theoretical research in this interesting scientific field.

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