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A NOTE ON THE EXTENDED LIENARD SYSTEM WITH DICKSON POLYNOMIALS OF THE THIRD KIND AS CORRECTIONS. THE LEVEL CURVES

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ABSTRACT: In this article a model with Dickson polynomials of the third kind as corrections in the Lienard differential system is presented.

The level curves are studied. The model is also considered in the light of Melnikov's approach. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, Dickson polynomials of the third kind as "correcting factors" in the Lienard system, level curves, Melnikov's approach.

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1. INTRODUCTION

Our previous results were related to the study of the class of Lienard polynomial systems of the type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= Poly_i(x) + \epsilon f_i(x) y \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$; $f_i(x)$ are specially chosen polynomials, and $Poly_i(x)$ are some of the known orthogonal polynomials.

In this article a model with Dickson polynomials of the third kind (see Fig. 1 c)) as corrections in the Lienard system is presented. The level curves are studied. The model is also considered in the light of Melnikov's approach. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS

Research can be successfully continued using other known classes of polynomials as corrections to the classical Lienard differential model. The Dickson polynomials of the third kind are given by (see for example [1]–[3])

$$D_{n,2}(x,a) := \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n-2i}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}; \;\; a \in R$$

$$\begin{cases} D_{1,2}(x,a) &= x \\ D_{3,2}(x,a) &= x^3 - ax \\ D_{5,2}(x,a) &= x^5 - 3ax^3 + a^2 x \\ D_{7,2}(x,a) &= x^7 - 5ax^5 + 6a^2x^3 - a^3 x \\ D_{9,2}(x,a) &= x^9 - 7ax^7 + 16a^2x^5 - 10a^3x^3 + a^4 x \end{cases}$$

Dickson polynomials of the first, second and third kind for fixed $a = 1$ are depicted in Fig. 1.

Let $Poly_i(x)$ (in (1)) coincides with Dickson polynomials $D_{n,2}(x)$. Without going into details, we will note some more interesting level curves:

The case 1): $n = 7$.

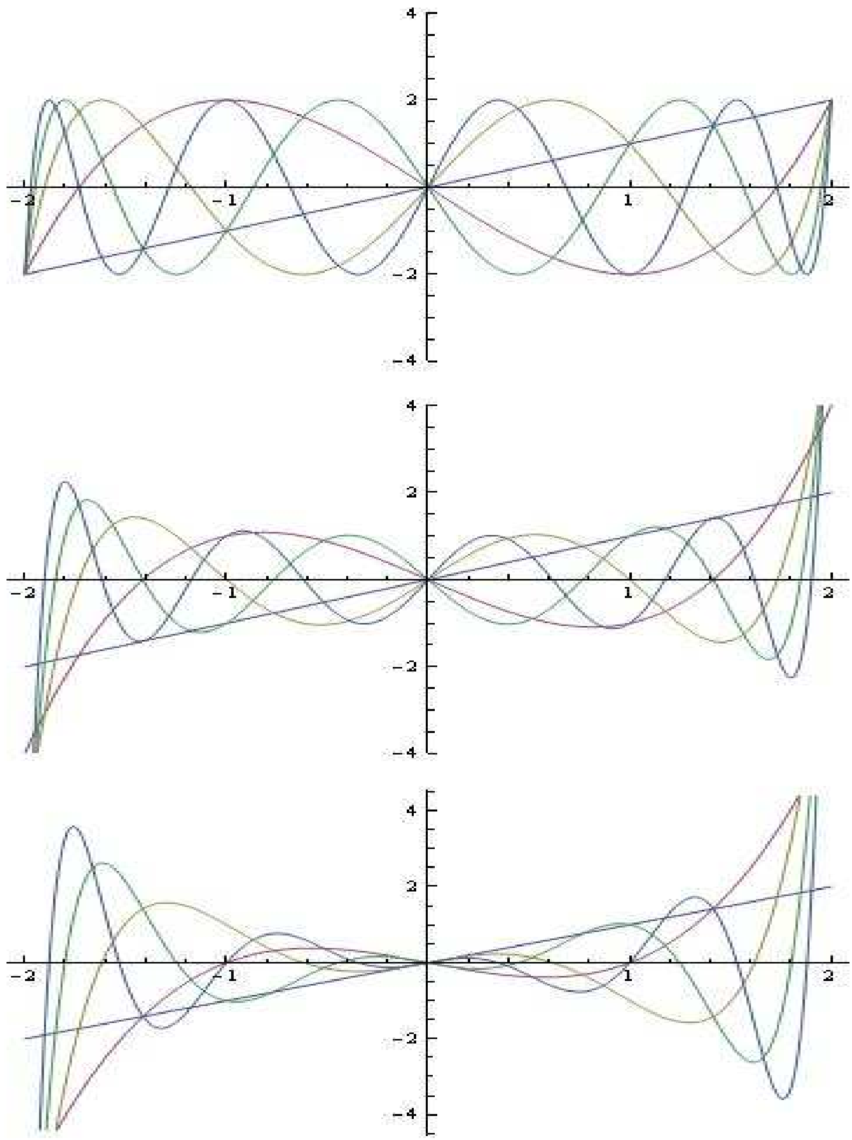

Figure 1: a) Dickson polynomials of the first kind; b) Dickson polynomials of the second kind; c) Dickson polynomials of the third kind.

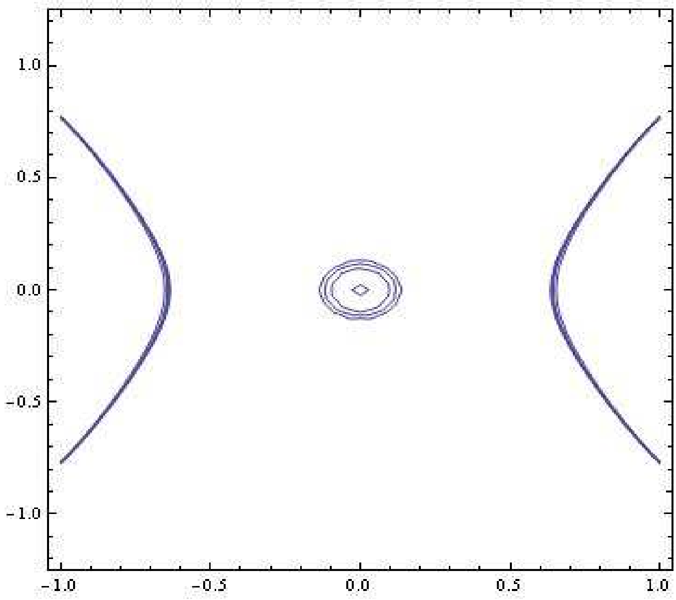

Figure 2: Level curves (the case 1).

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{8}x^8 + \frac{5}{6}x^6 - \frac{3}{2}x^4 + \frac{1}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 2.

The case 2): $n = 9$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{10}x^{10} + \frac{7}{8}x^8 - \frac{5}{2}x^6 + \frac{5}{2}x^4 - \frac{1}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 3.

The case 3): $n = 11$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{12}x^{12} + \frac{9}{10}x^{10} - \frac{7}{2}x^8 + \frac{35}{6}x^6 - \frac{15}{4}x^4 + \frac{1}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 4.

Some simulations

The simulations on the Lienard–type systems:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -D_{7,2}(x) + \epsilon f(x) y \end{cases} \tag{2}$$

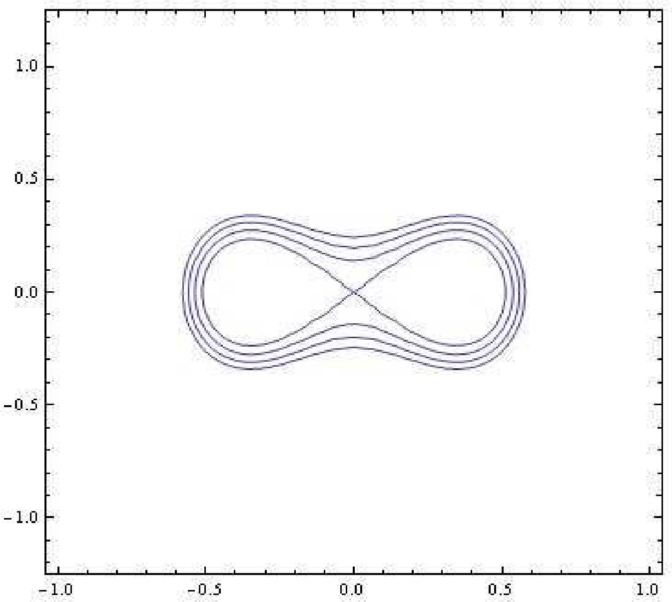

Figure 3: Level curves (the case 2).

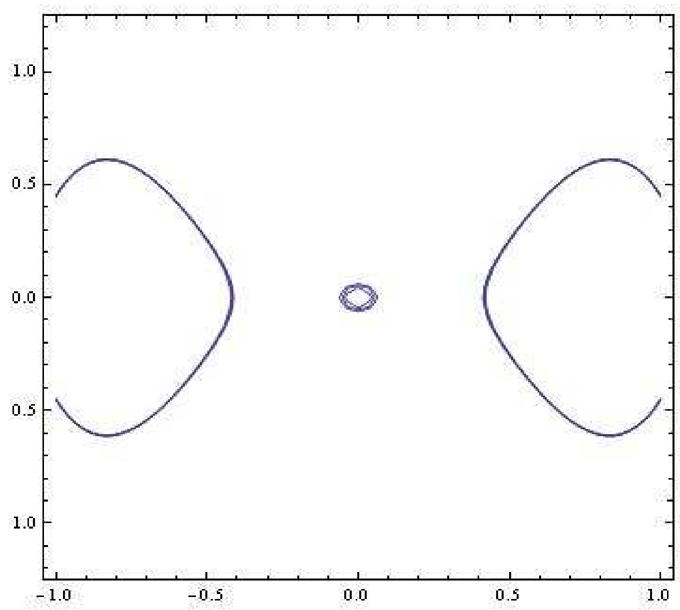

Figure 4: Level curves (the case 3).

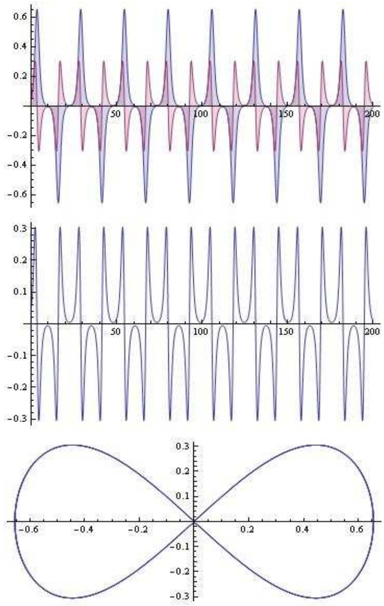

Figure 5: The simulations (system (2)) for $x_0 = 0.05; y_0 = 0.05; \epsilon = 0.0001$).

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -D_{9,2}(x) + \epsilon f(x) y \end{cases} \tag{3}$$

where $f(x) = x - \frac{1}{3}x^3$ is the Van der Poll "correction" are depicted on Fig. 5–Fig. 6.

The new model in the light of Melnikov's considerations

The following result provides the necessary information about the number of limit cycles and their radii [7]–[8]:

The Lienard system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon \left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1} \right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree Melnikov polynomial [5] in r^2,

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 r^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} r^{2n}$$

has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

The case $n = 9$.

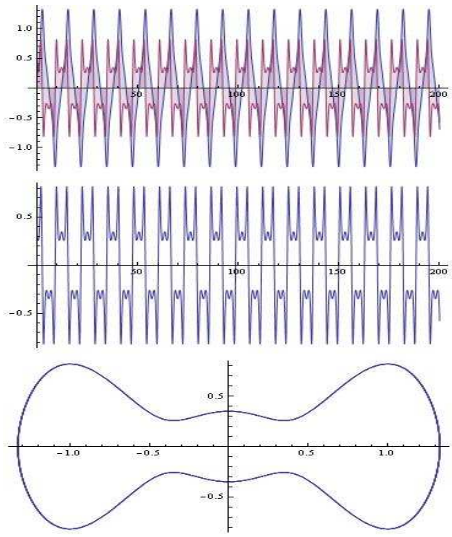

Figure 6: The simulations (system (3) for $x_0 = 0.2; y_0 = 0.3; \epsilon = 0.0001$).

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(\mu x - 10x^3 + 15x^5 - 7x^7 + x^9) \\ \dfrac{dy}{dt} &= -x \end{cases} \quad (5)$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 1. The Lienard–type system for $n = 9$, and for all sufficiently small $\epsilon \neq 0$

a) for $\mu = 1$ has four simple limit cycles;

b) for $\mu = 1.9$ has limit cycles: 0.7771001 (with multiplicity – two) and 1.81568 (with multiplicity – two).

Proof. For the Melnikov polynomial in r^2 (see Fig. 7) we have:

$$P(r^2, 4) = \frac{\mu}{2} - \frac{15}{4}r^2 + \frac{75}{16}r^4 - \frac{245}{128}r^6 + \frac{63}{256}r^8. \quad (6)$$

Evidently, for example $\mu = 1$ we have four simple limit cycles: 0.406438, 1.11846, 1.62678 and 1.9275 and for $\mu = 1.9$ has limit cycles: 0.7771001 and 1.81568 with multiplicity – two.

The case $n = 11$.

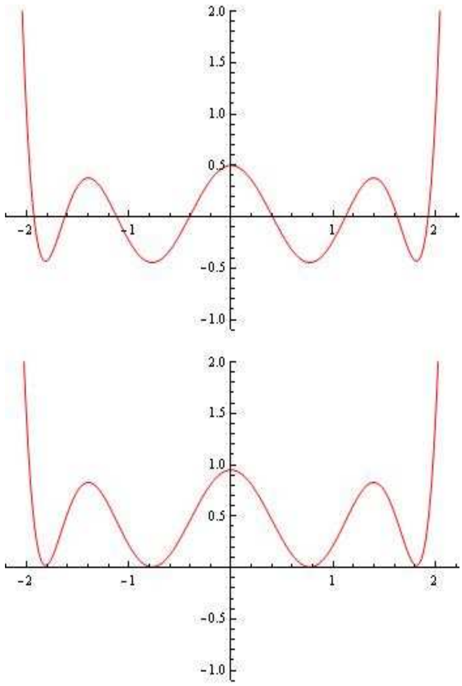

Figure 7: The Melnikov polynomial $P(r^2, 4)$ for $n = 9$ ($D_9(x, a)$; $a = 1$): a) $\mu = 1$ (four simple limit cycles); b) $\mu = 1.9$ (two limit cycles with multiplicity – two.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(-\mu x + 15x^3 - 35x^5 + 28x^7 - 9x^9 + x^{11}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{7}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 2. The Lienard–type system for $n = 11$, and for all sufficiently small $\epsilon \neq 0$

a) for $\mu = 1$ has four simple limit cycles: 0.333274, 0.932905, 1.40221, 1.74969 and 1.95172;

b) for $\mu = 1.694899$ has simple limit cycles: 0.509618, 0.75707, 1.97313 and 1.59563 (with multiplicity – two);

c) for $\mu = 0.2999999$ has simple limit cycles: 0.167871, 1.83587, 1.91053 and 1.17678 (with multiplicity – two).

Proof. For the Melnikov polynomial in r^2 (see Fig. 8) we have:

$$P(r^2, 5) = \frac{-\mu}{2} + \frac{45}{8}r^2 - \frac{175}{16}r^4 + \frac{245}{32}r^6 - \frac{567}{256}r^8 + \frac{231}{1024}r^{10}. \tag{8}$$

Evidently, for example $\mu = 1$ we have four simple limit cycles: 0.406438, 1.11846, 1.62678 and 1.9275; for $\mu = 1.9$ – simple limit cycles: 0.7771001 and 1.81568 with multiplicity – two; for $\mu = 0.2999999$ (see Fig. 9) – simple limit cycles: 0.167871, 1.83587, 1.91053 and 1.17678 with multiplicity – two.

Remarks.

1. To determine the zeros of the Melnikov polynomial, a computational procedure described in detail in [23] is essentially used.

2. For $\mu = 1.9$ (the case $n = 9$; see Proposition 1) the Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{10}x^{10} + \frac{7}{8}x^8 - \frac{5}{2}x^6 + \frac{5}{2}x^4 - \frac{\mu}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 10.

3. For $\mu = 1.694899$ (the case $n = 11$; see Proposition 2) the Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x, y) = \frac{y^2}{2} - \frac{1}{12}x^{12} + \frac{9}{10}x^{10} - \frac{7}{2}x^8 + \frac{35}{6}x^6 - \frac{15}{4}x^4 + \frac{\mu}{2}x^2.$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 11.

4. For $\mu = 0.2999999$ the level curves $L_{h_i} = \{H(x, y) = h_i\}$ are depicted at Fig. 12.

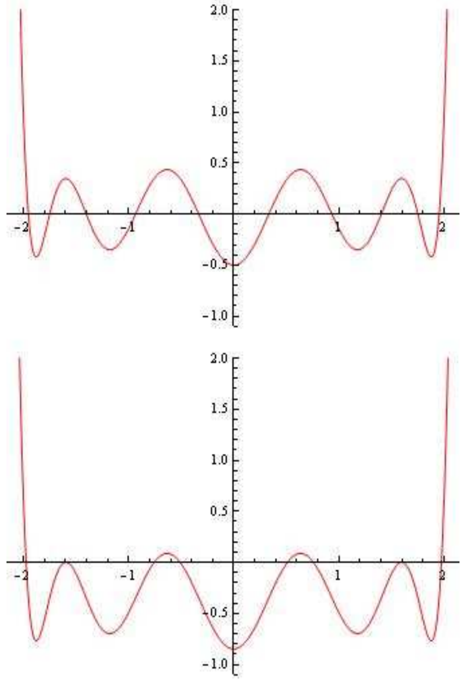

Figure 8: The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ ($D_{11}(x, a)$; $a = 1$): a) $\mu = 1$ (five simple limit cycles); b) $\mu = 1.694899$ (three simple limit cycles and limit cycle with multiplicity – two.

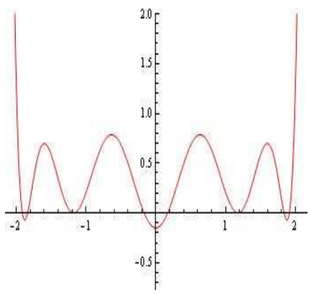

Figure 9: The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ ($D_{11}(x, a)$; $a = 1$): c) for $\mu = 0.2999999$ has simple limit cycles: 0.167871, 1.83587, 1.91053 and 1.17678 with multiplicity – two.

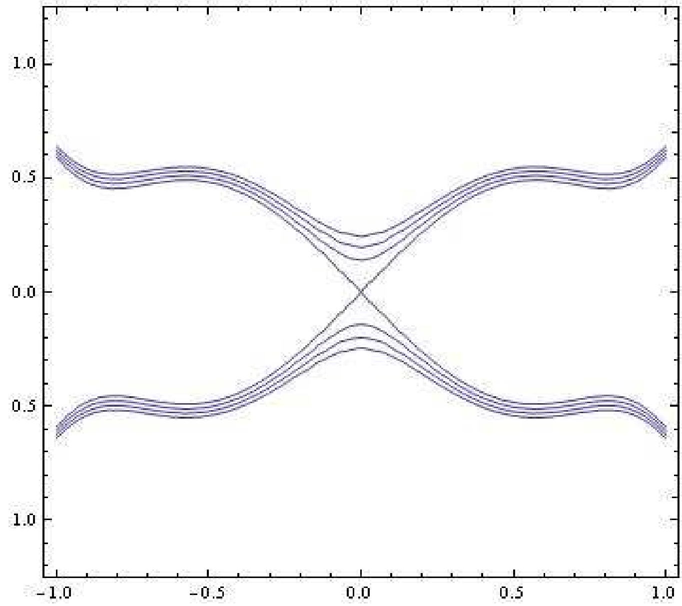

Figure 10: Level curves for $\mu = 1.9$.

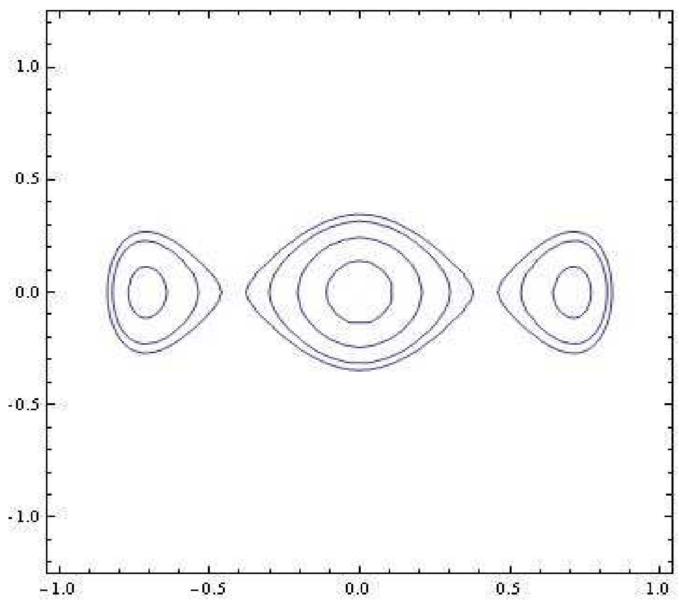

Figure 11: Level curves for $\mu = 1.694899$.

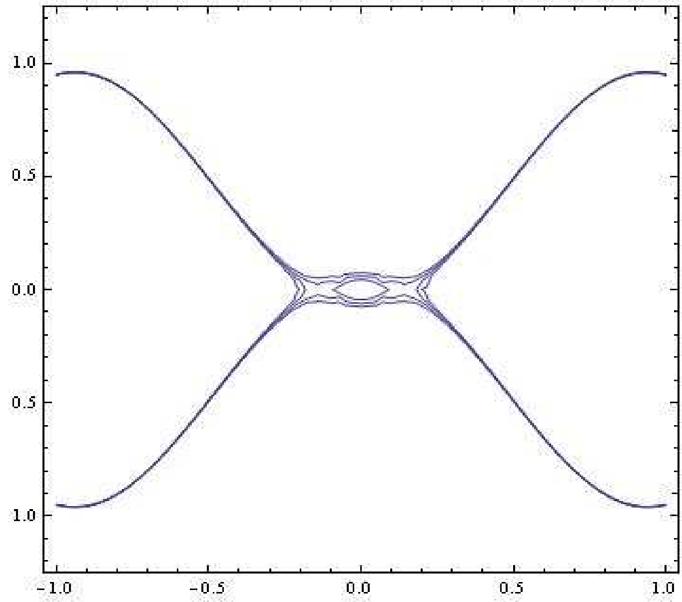

Figure 12: Level curves for $\mu = 0.2999999$.

We note that in a number of cases, the component $y(t)$ of the solution can be used for modeling and synthesis of "U and V"–shaped transfer functions and electric circuits.

5. Consider the following model in the light of Zeeman's approach:

$$\begin{cases} \dfrac{dx}{dt} &= c(F(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{9}$$

with $c > 0$ and

$$F(x) = -px + 15x^3 - 35x^5 + 28x^7 - 9x^9 + x^{11}$$

The catastrophe surfaces $(x, y, p) = F(x) - y$ $(p = 5, 10, 20)$ for the model is depicted on Fig. 13.

6. The Dickson polynomials of the $(m + 1)$–th kind are given by (see for example [24])

$$D_{n,m}(x, a) := \sum_{i=0}^{[\frac{n}{2}]} \frac{n - mi}{n - i} \begin{pmatrix} n - i \\ i \end{pmatrix} (-a)^i x^{n-2i}; \;\; a \in R.$$

Polynomials of this type can be used also as correction factors in some differential models.

For other results see [9]–[22].

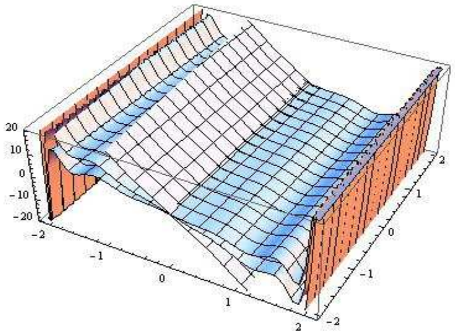

Figure 13: The catastrophe surfaces in the light of Zeeman considerations.

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