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ASSOCIATED HERMITE POLYNOMIALS. SOME APPLICATIONS

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ABSTRACT: In this article we will consider a class of associated Hermite polynomials generated by recurrence formula $H^a_{n+1}(x) = 2xH^a_n(x) - 2(n + a)H^a_{n-1}(x)$; $H^a_0(x) = 1$; $H^a_1(x) = 2x$ with possible applications in two directions – the study the dynamics of differential systems and the generation of special classes of radiation diagrams.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, associated Hermite polynomials $H^a_{n+1}(x)$, Melnikov's approach, level curves, emitting chart.

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1. INTRODUCTION

The associated Hermite polynomials are a sequence of orthogonal polynomials considered by Askey and Wimp [1]. Consider the polynomials defined as:

$$H^a_{n+1}(x) = 2xH^a_n(x) - 2(n + a)H^a_{n-1}(x)$$

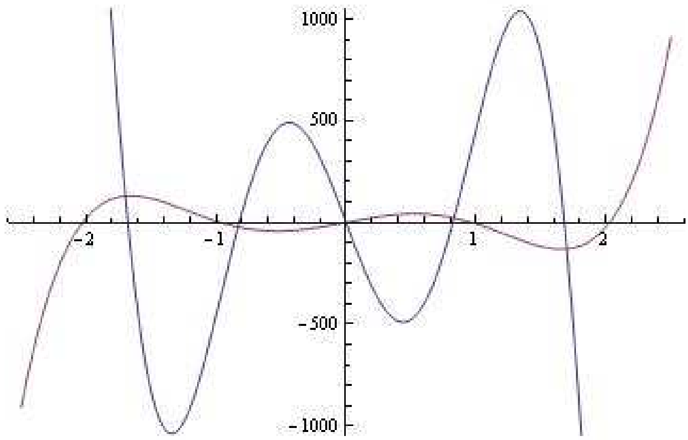

Figure 1: The polynomials $H_n^a(x)$ for fixed $a = 0.02$; $n = 5, 7$.

with

$$H_0^a(x) = 1; \;\; H_1^a(x) = 2x.$$

Then we have (see Fig. 1 for fixed $a = 0.02$; $n = 5, 7$)

$$\left\{ \begin{array}{rcl} H_3^a(x) & = & 8x^3 - 4(3+2a)x \\ H_5^a(x) & = & 32x^5 - 32(2a+5)x^3 + 24(a^2+5a+5)x \\ H_7^a(x) & = & 128x^7 - 192(7+2a)x^5 + 160(21+2a^2+14a)x^3 - \\ & & -16(105+128a+42a^2+4a^3)x \\ H_9^a(x) & = & 512x^9 - 1024(9+2a)x^7 + 2688(18+a^2+9a)x^5 - \\ & & -640(126+2a^3+27a^2+109a)x^3 + 160(189+ \\ & & +a^4+18a^3+110a^2+261a)x \end{array} \right.$$

Consider the Lienard system

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & Poly(x) + \epsilon F(x)y \end{array} \right. \tag{1}$$

where $0 \le \epsilon < 1$; $F(x)$ are specially chosen polynomials, and $Poly(x)$ are the polynomials $H_n^a(x)$.

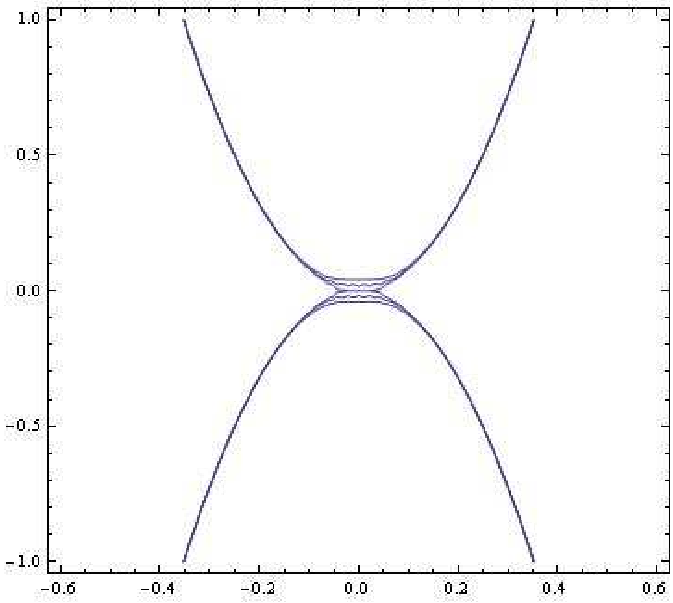

Figure 2: Level curves (the case 1).

2. MAIN RESULTS

Without going into details, we will note some more interesting level curves.

2.1. THE LEVEL CURVES

The case 1): $n = 5$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{16}{3}x^6 + 8(2a+5)x^4 - 12(a^2+5a+5)x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.25$ are depicted at Fig. 2.

The case 2): $n = 7$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$\begin{aligned} H(x,y) &= \tfrac{y^2}{2} - 16x^8 + 32(7+2a)x^6 - 40(21+2a^2+14a)x^4 + 8(195+ \\ &\quad +8a + 42a^2 + 4a^3)x^2. \end{aligned}$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.04$ are depicted at Fig. 3.

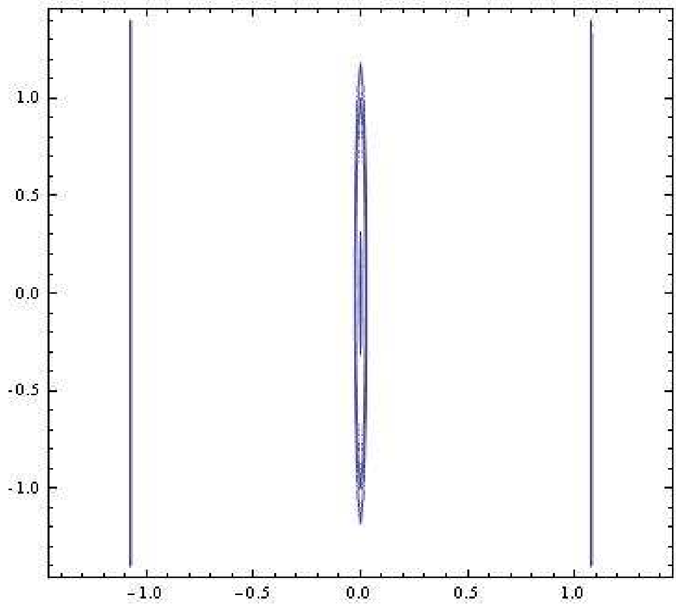

Figure 3: Level curves (the case 2).

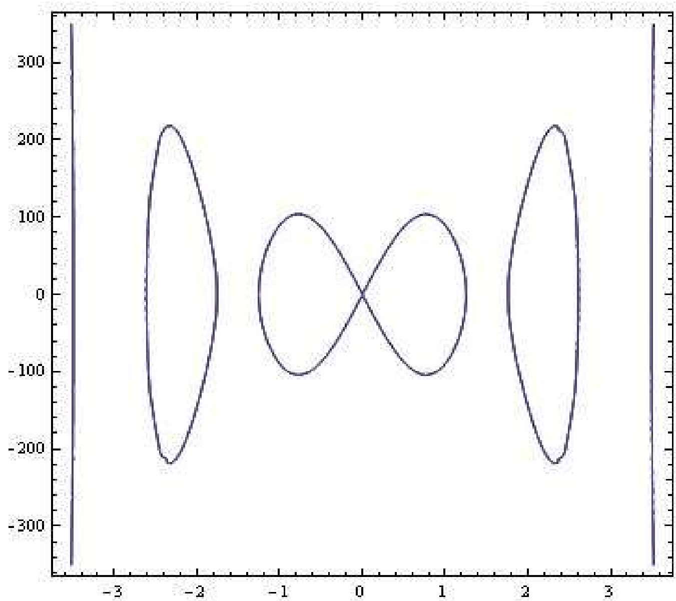

Figure 4: Level curves (the case 3).

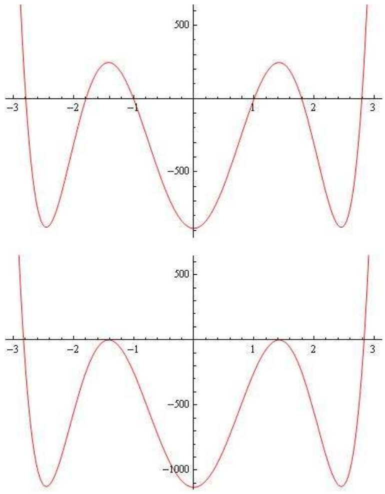

Figure 5: a) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$; $a = 0.01$ and $\mu = 1770.55$ (three limit cycles); b) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$; $a = 0.01$ and $\mu = 2264.0789309$ (simple limit cycle 2.83146 and limit cycle 1.41724 with multiplicity – two).

The case 3): $n = 9$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$\begin{aligned} H(x, y) &= \tfrac{y^2}{2} - \tfrac{256}{5}x^{10} + 128(9 + 2a)x^8 - 448(18 + a^2 + 9a)x^6 + \\ &+160(126 + 2a^3 + 27a^2 + 109a)x^4 - 80(189 + a^4 + \\ &+18a^3 + 110a^2 + 261a)x^2. \end{aligned}$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ for fixed $a = 0.04$ are depicted at Fig. 4.

2.2. THE MODEL IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS

The case $n = 7$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(128x^7 - 192(7 + 2a)x^5 + 160(21 + 2a^2 + 14a)x^3 - \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases} \qquad (2)$$

where $\mu > 0$, $\epsilon > 0$.

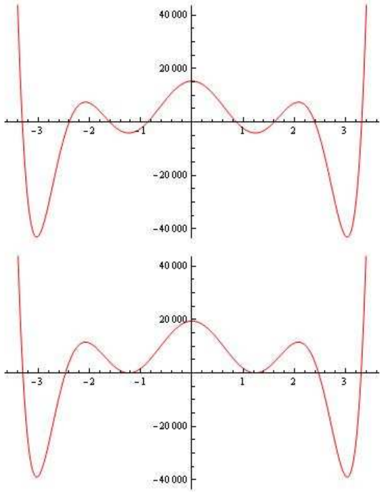

Figure 6: a) The Melnikov polynomial $M(r^2,4)$ for $n=9$; $a=0.01$; $\mu=30659.4$ (four limit cycles); b) The Melnikov polynomial $M(r^2,4)$ for $n=9$; $a=0.01$ and $\mu=38904.46987$ (simple limit cycles 2.47005, 3.30082 and limit cycle 1.23449 with multiplicity – two).

The following is valid

Proposition 1. The Lienard–type system for $n=7$, $a=0.01$ and for all sufficiently small $\epsilon \neq 0$

a) for $\mu = 1770.55$ has three hyperbolic limit cycles 1.00289, 1.79472 and 2.79419.

b) for $\mu = 2264.0789309$ has a simple limit cycle 2.83146 and limit cycle 1.41724 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 5) we have:

$$M(r^2,3) = -\frac{\mu}{2} + 60(21+2a^2+14a)r^2 - 60(7+2a)r^4 + 35r^6. \qquad (3)$$

Evidently, for example $\mu = 2264.0789309$ we have a simple limit cycle and cycle with multiplicity – two.

The case $n=9$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(512x^9 - 1024(9+2a)x^7 + 2688(18+a^2+9a)x^5 - \\ & \quad -640(126+2a^3+27a^2+109a)x^3 + \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 2. The Lienard–type system for $n = 9$, $a = 0.01$ and for all sufficiently small $\epsilon \neq 0$

a) for $\mu = 30659.4$ has four limit cycles 0.864797, 1.60665, 2.39678 and 3.3122.

b) for $\mu = 38904.46987$ has simple limit cycles 2.47005, 3.30082 and limit cycle 1.23449 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 6) we have:

$$M(r^2, 4) = \tfrac{\mu}{2} - 240(126 + 2a^3 + 27a^2 + 109a)r^2 + 840(18 + a^2 + 9a)r^4 - 180(9 + 2a)r^6 + 126r^8.$$

Evidently, for example $\mu = 38904.46987$ we have two simple limit cycles and cycle with multiplicity – two.

2.3. SOME SIMULATIONS

The simulation on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -H_5^a(x) + \epsilon F(x) y \end{cases} \tag{4}$$

where

$$F(x) = x - x^3 + x^5 - \frac{1}{7}x^7$$

with $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.11$, $b = 0.99$, $c = 1.9$ is depicted on Fig. 7–Fig. 8.

The simulation on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -H_7^a(x) + \epsilon F(x) y \end{cases} \tag{5}$$

where

$$F(x) = x - x^3 + x^5 - \frac{1}{7}x^7$$

with $x_0 = 1.5$, $y_0 = 1.3$, $a = 0.01$, $b = 0.39$, $c = 1.5$ is depicted on Fig. 9–Fig. 10.

We will note that some specifics of the amplitudes of these polynomials open up the possibility of modeling signals from the field of antenna-feeder technology.

It is easy to take into account that the change of the variable t with $t = b\cos\theta + c$ (θ is the azimuthal angle and c is the phase difference) in the $y(t)$-component of the solution of the systems (4)–(5) leads to radiation diagrams.

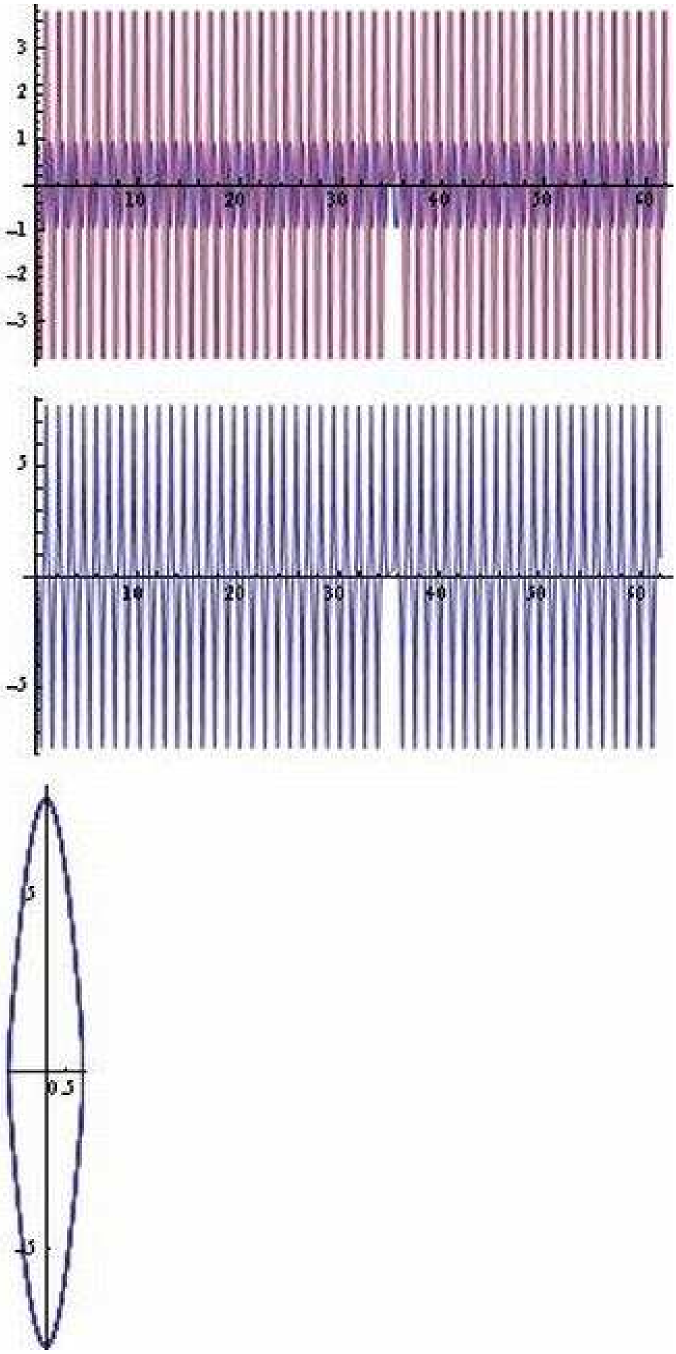

Figure 7: The simulations (system (4)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.11$, $b = 0.99$, $c = 1.9$; $\epsilon = 0.0001$: a) the solution of the system; b) y-component of the solution; c) the portrait.

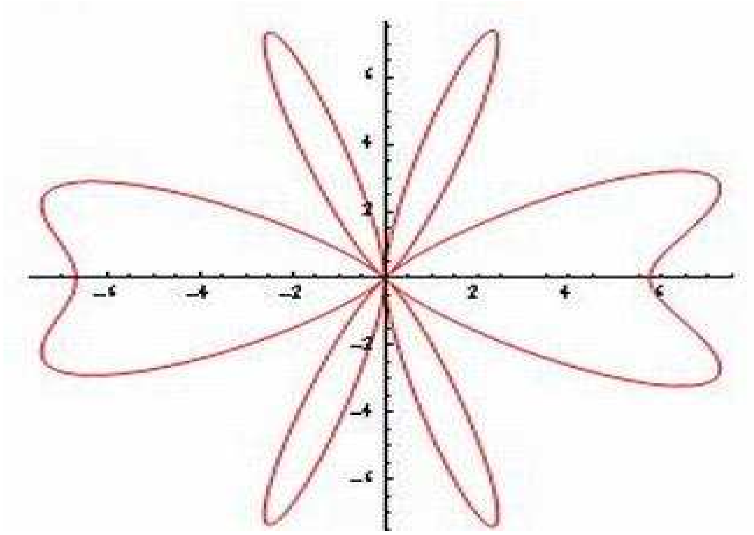

Figure 8: The simulations (system (4)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.11$, $b = 0.99$, $c = 1.9$; $\epsilon = 0.0001$: d) emitting chart.

2.4. THE MODEL IN THE LIGHT OF ZEEMAN'S CONSIDERATIONS

Consider the following model in the light of Zeeman's approach:

$$\begin{cases} \dfrac{dx}{dt} &= c(G(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{6}$$

with $c > 0$ and

$$G(x) = 128x^7 - 192(7 + 2a)x^5 + 160(21 + 2a^2 + 14a)x^3 - qx$$

The catastrophe surfaces $(x, y, p) = G(x) - y$ $(q = 500, 1000, 2000, 3000)$ for the model is depicted on Fig. 11.

Consider the model:

$$\begin{cases} \dfrac{dx}{dt} &= c(G(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases}$$

with $c > 0$ and

$$G(x) = 512x^9 - 1024(9+2a)x^7 + 2688(18+a^2+9a)x^5 - 640(126+2a^3+27a^2+109a)x^3 + qx$$

The catastrophe surfaces $(x, y, p) = G(x) - y$ $(q = 3000, 5000, 7500, 10000)$ for the model is depicted on Fig. 12.

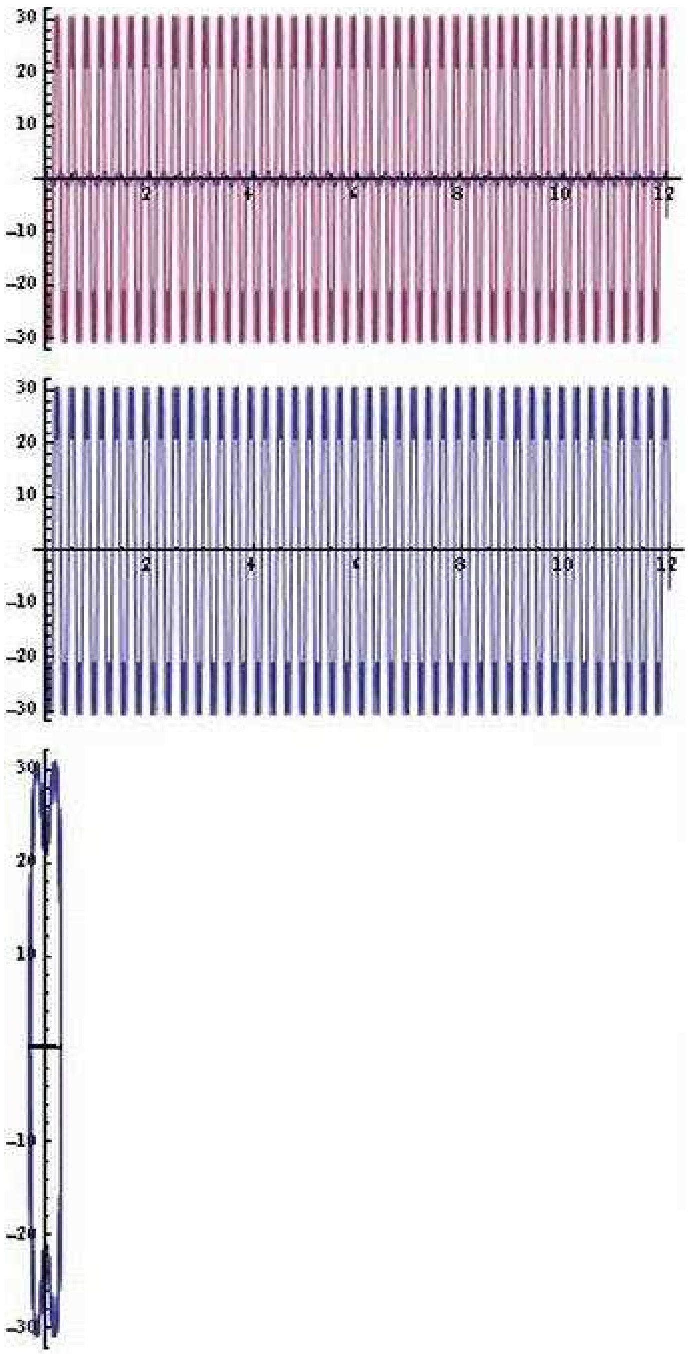

Figure 9: The simulations (system (5)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.11$, $b = 0.99$, $c = 1.9$; $\epsilon = 0.0001$: a) the solution of the system; b) y-component of the solution; c) the portrait.

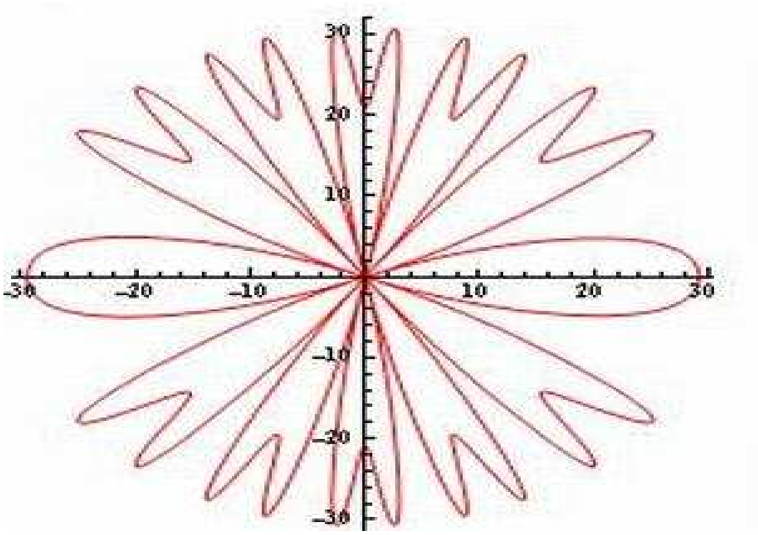

Figure 10: The simulations (system (5)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.11$, $b = 0.99$, $c = 1.9$; $\epsilon = 0.0001$: d) emitting chart.

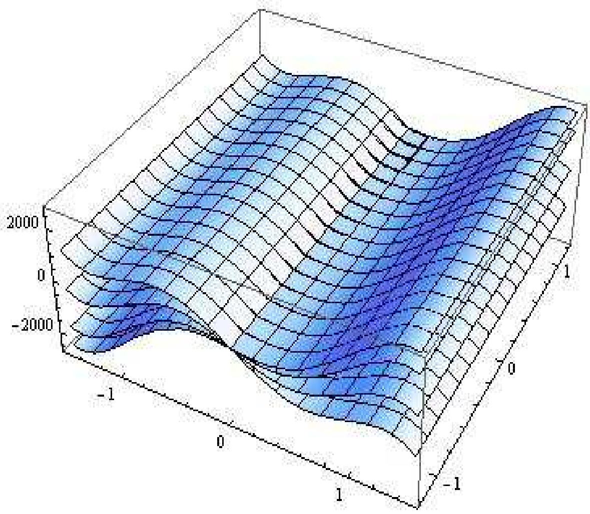

Figure 11: The catastrophe surfaces in the light of Zeeman considerations.

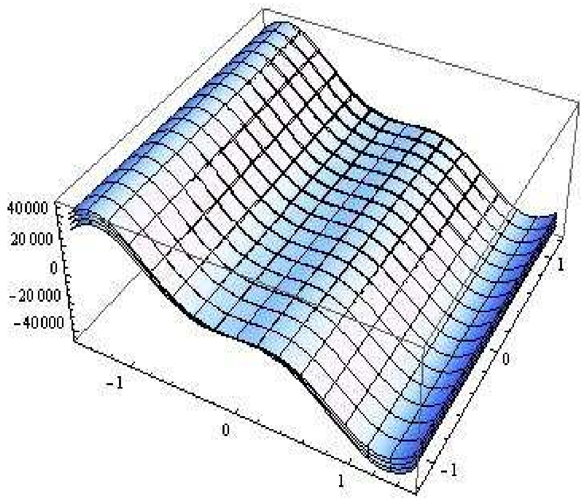

Figure 12: The catastrophe surfaces in the light of Zeeman considerations.

2.5. REMARKS.

In a number of cases, some interesting classes of diagram multipliers of radiating antennas can be modeled using the intrinsic properties of the associated Hermite polynomials (more precisely their amplitudes) as "corrections" to the differential Lienard system, and of course with a suitable choice of the function F.

As an example, we will consider the differential system with correction factors $H_9^a(x)$ and $F(x) = x - \frac{1}{3}x^3$ (Van der Poll correction).

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & -H_9^a(x) + \epsilon F(x) y \end{cases}$$

As a result of the simulation on the y–component of the solution of this system (with fixed $x_0 = 1.1, y_0 = 0.32, a = 0.1, b = 1.1, c = -1.1$)

$$y = \frac{|y(b\cos\theta + c|}{0.53 \times 10^{14}}$$

we get a sectional diagram model (with restriction) see Fig. 13 c).

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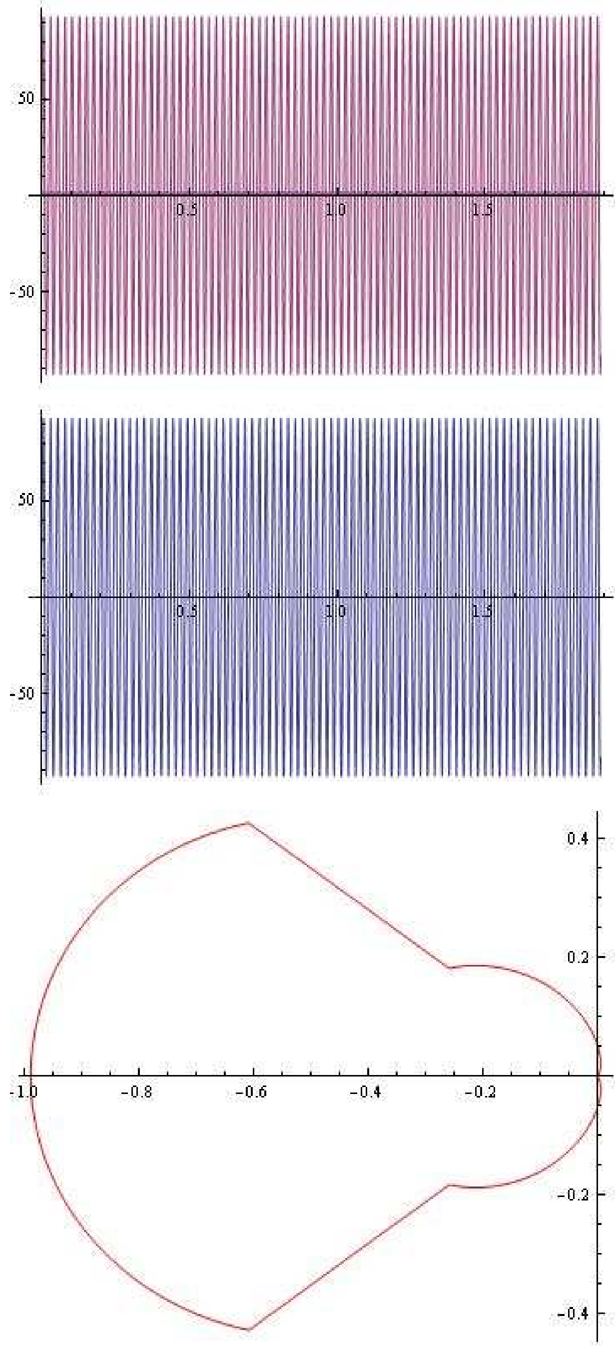

Figure 13: The simulations for $x_0 = 1.1, y_0 = 0.32, a = 0.1, b = 1.1, c = -1.1$; $\epsilon = 0.0001$: a) the solution of the system; b) y-component of the solution; c) sectional diagram model (with restriction).

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