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ASSOCIATED GEGENBAUER POLYNOMIALS. APPLICATIONS

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ABSTRACT: In this article we will consider a class of associated Gegenbauer polynomials generated by recurrence formula $V_{n+1}^a(x) = 2\frac{n+a}{n+1}xV_n^a(x) - \frac{n+2a-1}{n+1}V_{n-1}^a(x)$; $V_0^a(x) = 0$; $V_1^a(x) = 1$ with possible applications in two directions – the study the dynamics of differential systems and the generation of special classes of radiation diagrams.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, associated Gegenbauer polynomials $V_{n+1}^a(x)$, level curves, emitting chart.

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1. INTRODUCTION

The associated Gegenbauer polynomials are a sequence of orthogonal polynomials defined as (see for example [1]):

$$V_{n+1}^a(x) = 2\frac{n+a}{n+1}xV_n^a(x) - \frac{n+2a-1}{n+1}V_{n-1}^a(x)$$

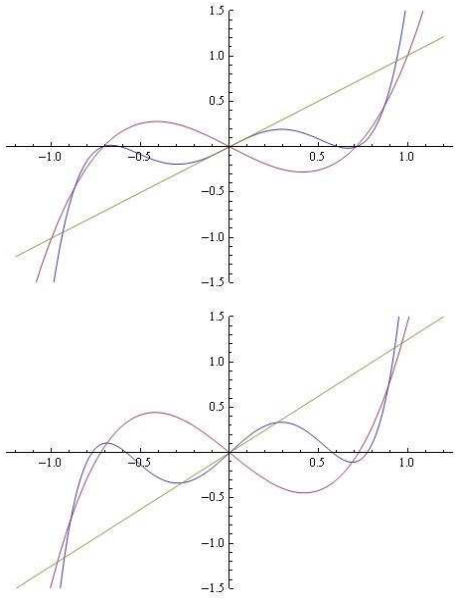


Figure 1: The polynomials $V_2^a(x), V_4^a(x)$ and $V_6^a(x)$; a) for $a = 0.01$; b) for $a = 0.25$.

with

$$V_0^a(x) = 0; \;\; V_1^a(x) = 1.$$

Then we have (see Fig. 1)

$$\left\{ \begin{array}{lcl} V_2^a(x) & = & (1+a)x \\ V_4^a(x) & = & \frac{1}{3}(1+a)(2+a)(3+a)x^3 - \frac{1}{6}(6+13a+5a^2)x \\ V_6^a(x) & = & \frac{2}{45}(1+a)(2+a)(3+a)(4+a)(5+a)x^5 - \\ & & -\frac{1}{45}(210+556a+444a^2+136a^3+14a^4)x^3 + \\ & & +\frac{1}{90}(90+244a+171a^2+33a^3)x \end{array} \right.$$

Consider the Lienard system

$$\left\{ \begin{array}{lcl} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & Poly(x) + \epsilon F(x) y \end{array} \right. \tag{1}$$

where $0 \leq \epsilon < 1$; $F(x)$ are specially chosen polynomials, and $Poly(x)$ are the polynomials $V_n^a(x)$.

Without going into details, we will note some more interesting level curves.

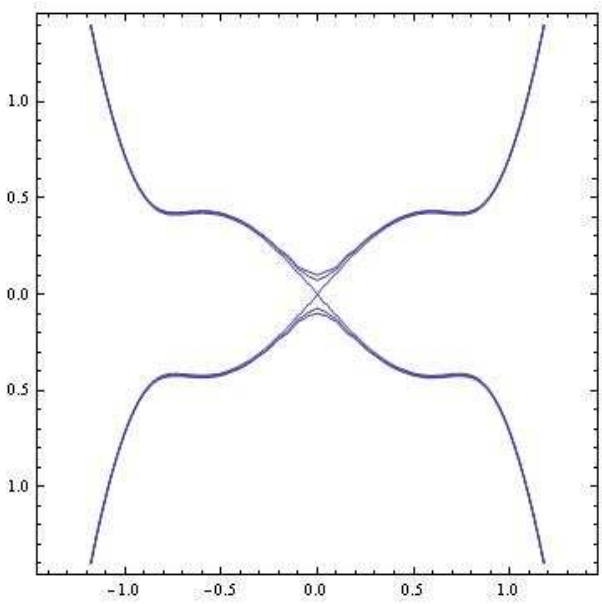


Figure 2: The level curves.

1.1. THE LEVEL CURVES

The case $n = 5$.

The Hamiltonian of system (1) ($\epsilon = 0$) is

$$\begin{aligned} H(x,y) &= \tfrac{y^2}{2} - \tfrac{1}{135}(1+a)(2+a)(3+a)(4+a)(5+a)x^6 + \\ &+ \tfrac{1}{180}(210 + 556a + 444a^2 + 136a^3 + 14a^4)x^4 - \\ &- \tfrac{1}{180}(90 + 244a + 171a^2 + 33a^3)x^2. \end{aligned}$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ for fixed $a = 0.1$ are depicted at Fig. 2.

1.2. SOME SIMULATIONS

The simulations on the Lienard–type system:

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} &=& y \\ \dfrac{dy}{dt} &=& -V_6^a(x) + \epsilon F(x) y \end{array} \right. \tag{2}$$

where

$$F(x) = x - x^3 + x^5 - \frac{1}{7}x^7$$

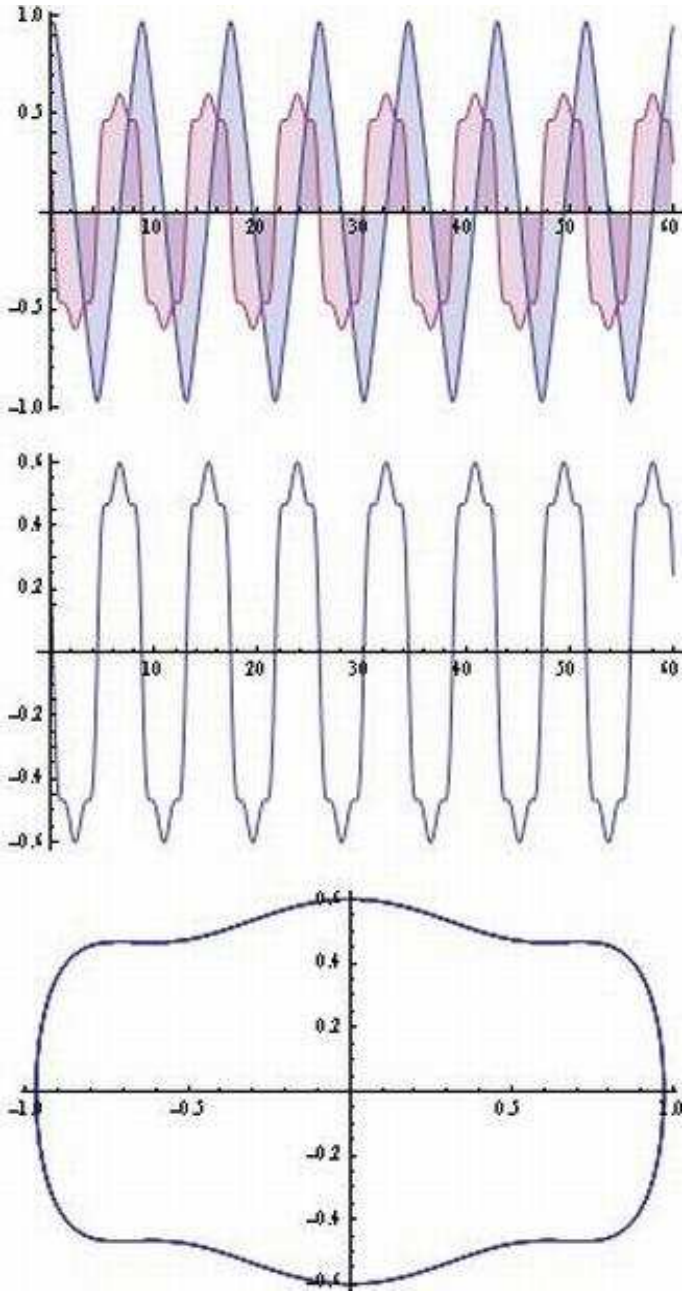


Figure 3: The simulations (system (2)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.01$, $b = -0.04$, $c = 0$; $\epsilon = 0.0001$: a) the solution of the system; b) y-component of the solution; c) the portrait.

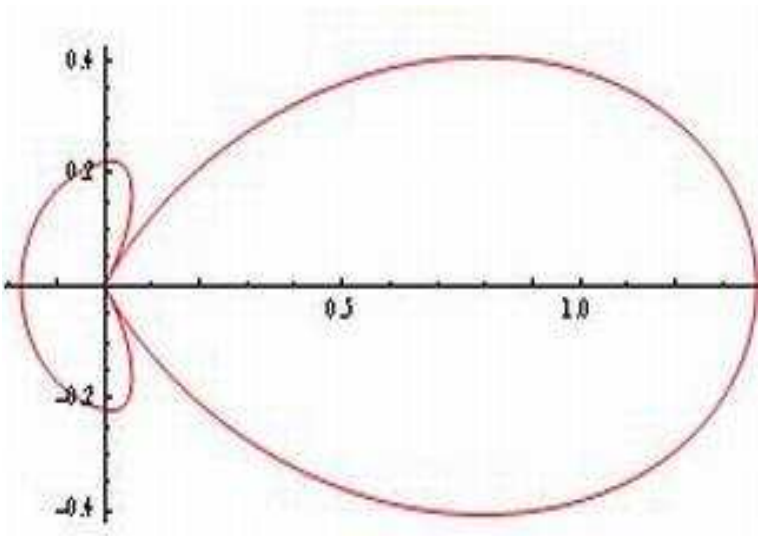


Figure 4: The simulations (system (2)) for $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.01$, $b = -0.04$, $c = 0$; $\epsilon = 0.0001$: d) emitting chart.

with

1) $x_0 = 0.95$, $y_0 = 0.22$, $a = 0.01$, $b = -0.04$, $c = 0$;

2) $x_0 = 0.99$, $y_0 = 0.2$, $a = 0.1$, $b = -0.1$, $c = 0.16$

are depicted on Fig. 3–Fig. 6.

1.3. THE MODEL IN THE LIGHT OF ZEEMAN'S CONSIDERATIONS

Consider the following model in the light of Zeeman's approach:

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & c(G(x) - y) \\ \dfrac{dy}{dt} & = & \frac{1}{c}x \end{array} \right. \tag{3}$$

with $c > 0$ and

$$\begin{aligned} G(x) &= \tfrac{2}{45}(1+a)(2+a)(3+a)(4+a)(5+a)x^5 - \\ &\quad -\tfrac{1}{45}(210 + 556a + 444a^2 + 136a^3 + 14a^4)x^3 + qx \end{aligned}$$

The catastrophe surfaces $(x, y, p) = G(x) - y$ $(q = 500, 1000, 2000, 3000)$ for the model is depicted on Fig. 7.

Acknowledgment

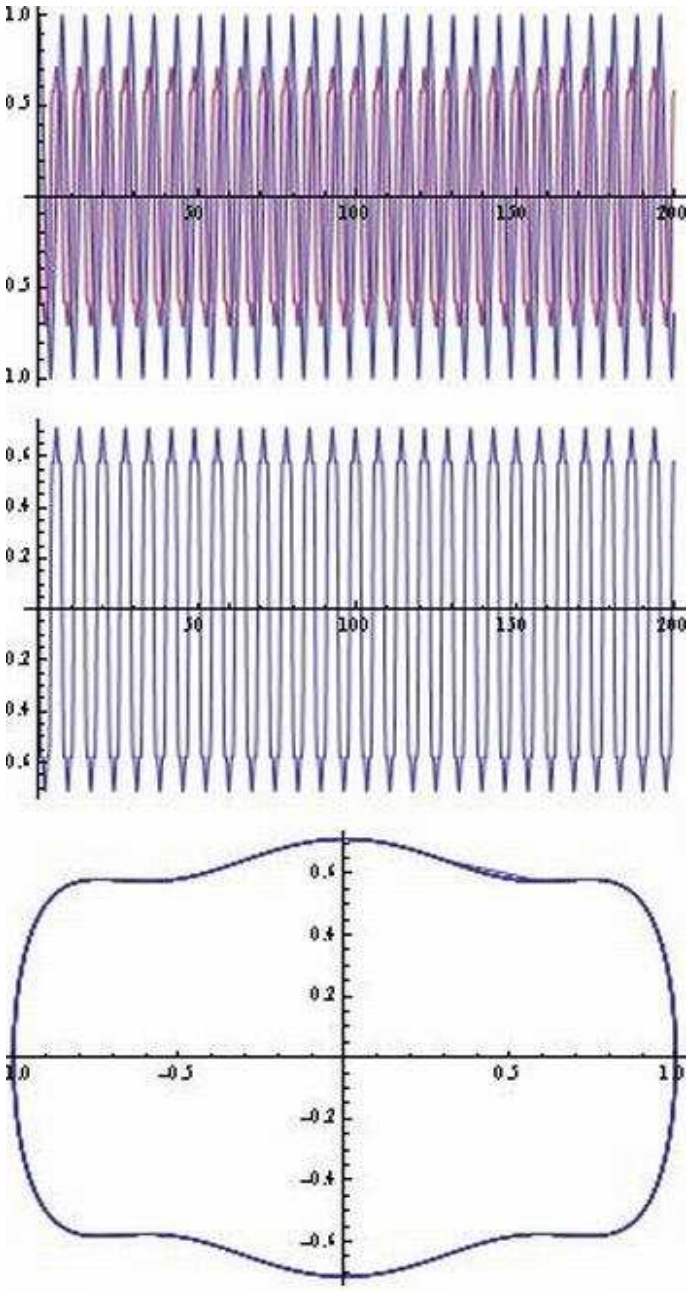


Figure 5: The simulations (system (2)) for $x_0 = 0.99$, $y_0 = 0.2$, $a = 0.1$, $b = -0.1$, $c = 0.16$; $\epsilon = 0.0001$: a) the solution of the system; b) y-component of the solution; c) the portrait.

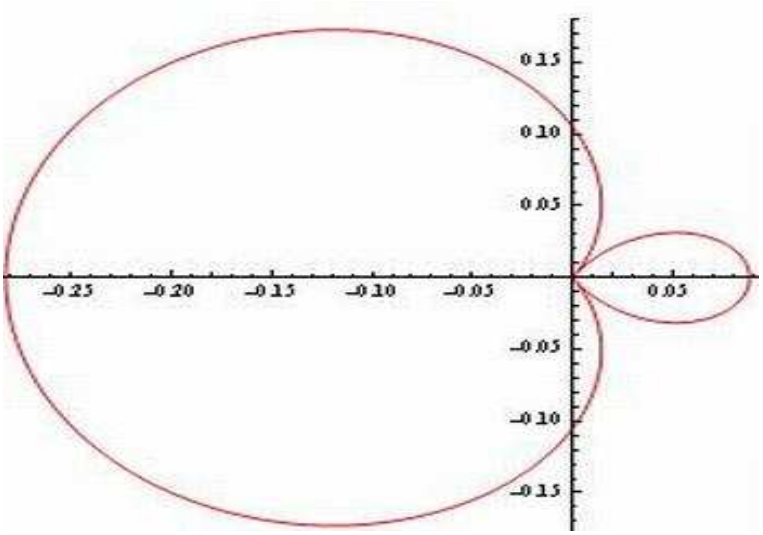


Figure 6: The simulations (system (2)) for $x_0 = 0.99$, $y_0 = 0.2$, $a = 0.1$, $b = -0.1$, $c = 0.16$; $\epsilon = 0.0001$: d) emitting chart.

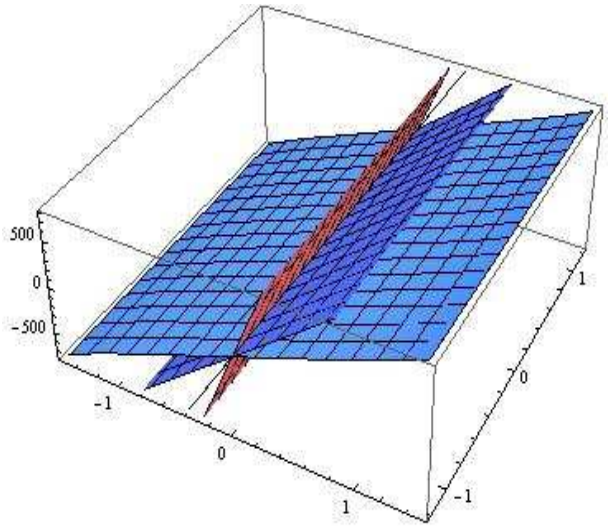


Figure 7: The catastrophe surfaces in the light of Zeeman considerations.

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