International Journal of Differential Equations and Applications
Volume **22**, No. 1 (2023), pages: 29-45
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

DYNAMICS OF THE LIENARD SYSTEM USING CONTINUOUS AND BIVARIATE *q*–HERMITE POLYNOMIALS. SOME APPLICATIONS

Maria Vasileva[1], Vesselin Kyurkchiev[1], Anton Iliev[1,2],
Asen Rahnev[1] and Nikolay Kyurkchiev[1,2]

[1]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

[2] Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, BULGARIA

ABSTRACT: In this article we consider continuous q–Hermite polynomials generated by relation $H_{m+1}(x;q) = 2xH_n(x;q) - (1-q^n)H_{n-1}(x;q)$; $H_0(x;q) = 1$; $H_{-1}(x;q) = 0$ and bivariate q–Hermite polynomials generated by relation $H_{m+1}(x,s,q) = xH_n(x,s,q) - (1-q^n)sH_{n-1}(x,s,q)$; $H_0(x,s,q) = 1$; $H_1(x,s,q) = x$ with possible applications in two directions – the study the dynamics of differential systems and the generation of special classes of radiation diagrams. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, continuous and bivariate q–Hermite polynomials, emitting chart

Received: April 28, 2023 **Revised:** June 5, 2023
Published: June 8, 2023 **doi:** 10.12732/ijdea.v22i1.3
Academic Publications, Ltd. https://acadpubl.eu

1. CONTINUOUS *Q*–HERMITE POLYNOMIALS

Continuous q–Hermite polynomials are at the bottom of a large class of q – hypergeometric polynomials to which most of their properties can be generalized [1]–[4].

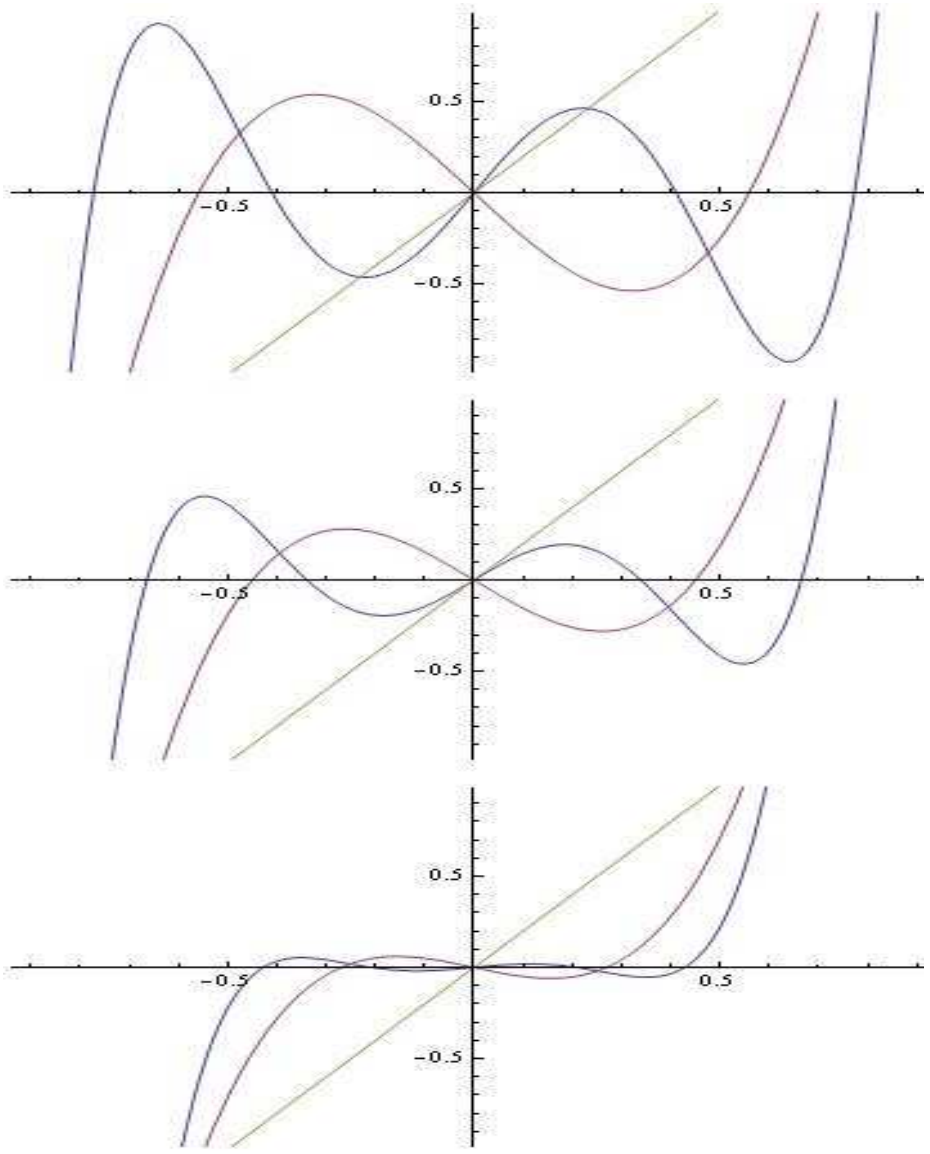


Figure 1: The continuous q–Hermite polynomials $H_1(x;q), H_3(x;q)$ and $H_5(x;q)$; a) $q = 0.5$; b) $q = 0.7$; $q = 0.9$.

An excellent overview of the subject can be found in [5].

The continuous q–Hermite satisfy the three–term recurrence relation

$$H_{m+1}(x;q) = 2xH_n(x;q) - (1-q^n)H_{n-1}(x;q)$$

with

$$H_0(x;q) = 1; \quad H_{-1}(x;q) = 0.$$

and $0 < q < 1$.

We have (see Fig. 1)

$$\begin{cases} H_1(x;q) &= 2x \\ H_3(x;q) &= 8x^3 - 2x(2-q-q^2) \\ H_5(x;q) &= 32x^5 - 8x^3(4-q-q^2-q^3-q^4)+ \\ & \quad +2x(3-2q-q^2-q^3-q^4+q^5+q^6) \end{cases}$$

1.1. SIMULATIONS

The simulation on the Lienard system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -H_5(x;q) + \epsilon F(x)y \end{cases} \tag{1}$$

where

$$F(x) = x - \frac{1}{3}x^3$$

for $q = 0.5$, $x_0 = 1.35$, $y_0 = 0.9$, $b = 1.05$, $c = 1.08$, $\epsilon = 0.0001$ is depicted on Fig. 2.

2. BIVARIATE Q–HERMITE POLYNOMIALS

The bivariate q–Hermite polynomials $H_n(x,s,q)$ satisfy the three–term recurrence relation

$$H_{m+1}(x,s,q) = xH_n(x,s,q) - (1-q^n)sH_{n-1}(x,s,q)$$

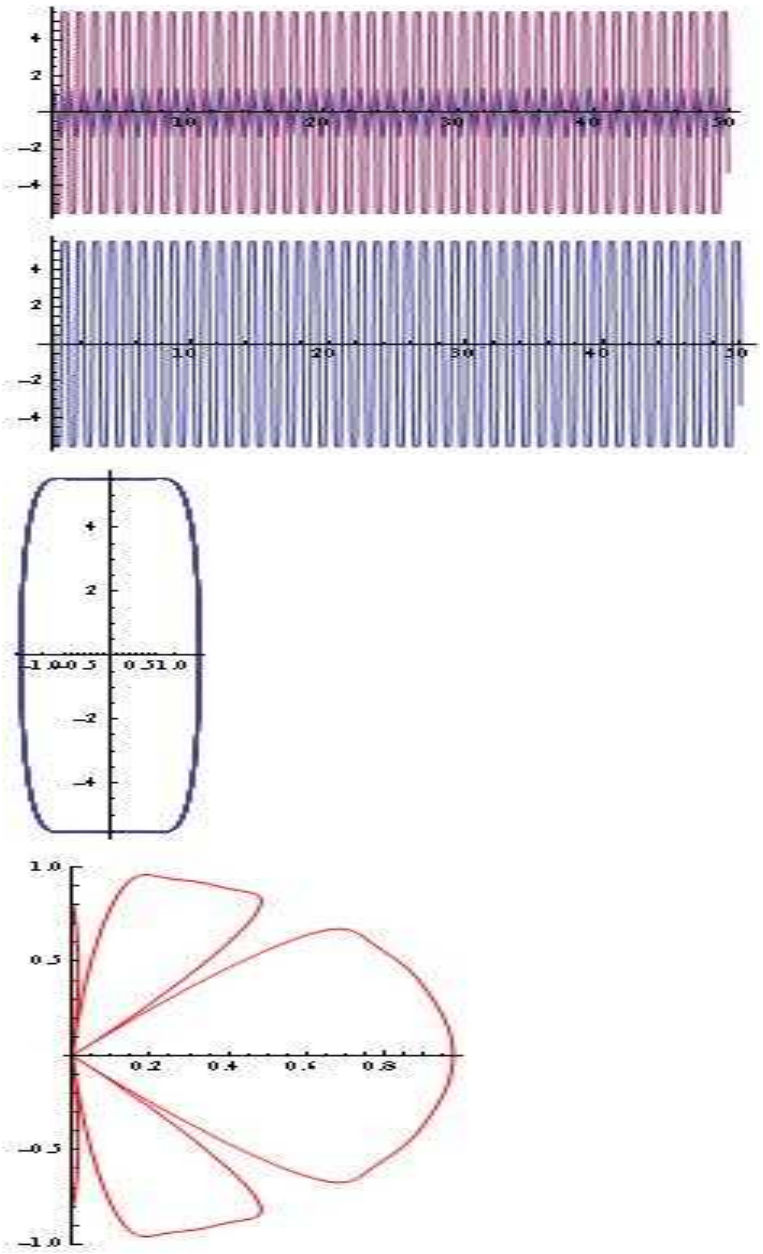


Figure 2: The simulations for $q = 0.5$, $x_0 = 1.35$, $y_0 = 0.9$, $b = 1.05$, $c = 1.08$, $\epsilon = 0.0001$: a) solutions of the system (1); b) y-component of the solution; c) the portrait; b) emitting chart (normalized).

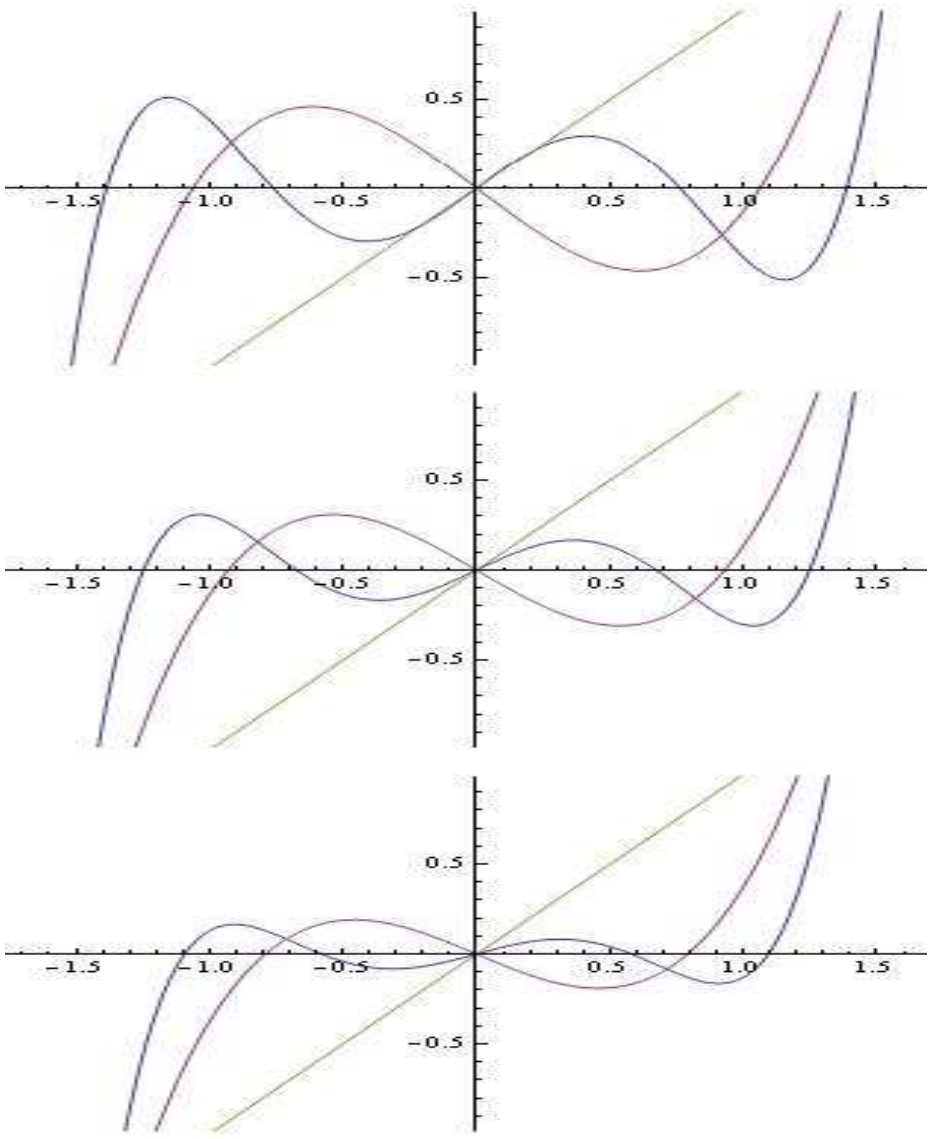


Figure 3: The bivariate q–Hermite polynomials $H_1(x, s, q), H_3(x, s, q)$ and $H_5(x, s, q)$; a) $q = 0.3$, $s = 0.7$; b) $q = 0.4$, $s = 0.6$; c) $q = 0.5$, $s = 0.5$.

with

$$H_0(x,s,q)=1;\quad H_1(x,s,q)=x.$$

and $0<q<1$.

Let q and s are parameters. We have (see Fig. 3)

$$\begin{cases} H_3(x,s,q) &= x^3-(1-q)(q+2)sx \\ H_5(x;q) &= x^5-sx^3(4-q-q^2-q^3-q^4)+ \\ & \quad +s^2(1-q)x(3+q-q^3-2q^4-q^5) \\ H_7(x,s,q) &= x^7-s(6-q-q^2-q^3-q^4-q^5-q^6)x^5 \\ & \quad +s^2(10-4q-3q^2-3q^3-2q^4-2q^5-2q^6+2q^7+2q^8+q^9+q^{10})x^3 \\ & \quad -s^3(1-q)(4+q-2q^3-2q^4-q^5-3q^6-q^7+q^9+2q^{10}+q^{11})x \end{cases}$$

2.1. SIMULATIONS

The simulation on the Lienard system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -H_5(x,s,q)+\epsilon F(x)y \end{cases} \tag{2}$$

where

$$F(x)=x-\frac{1}{3}x^3$$

for $q=0.5$, $s=0.5$ $x_0=0.45$, $y_0=0.033$, $b=0.75$, $c=0.9$, $\epsilon=0.0001$ is depicted on Fig. 4.

The simulation on the Lienard system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -H_7(x,s,q)+\epsilon F(x)y \end{cases} \tag{3}$$

where

$$F(x)=x-\frac{1}{3}x^3$$

for $q=0.5$, $s=0.5$ $x_0=1.01$, $y_0=9$, $b=0.99$, $c=0.9$, $\epsilon=0.0001$ is depicted on Fig. 5.

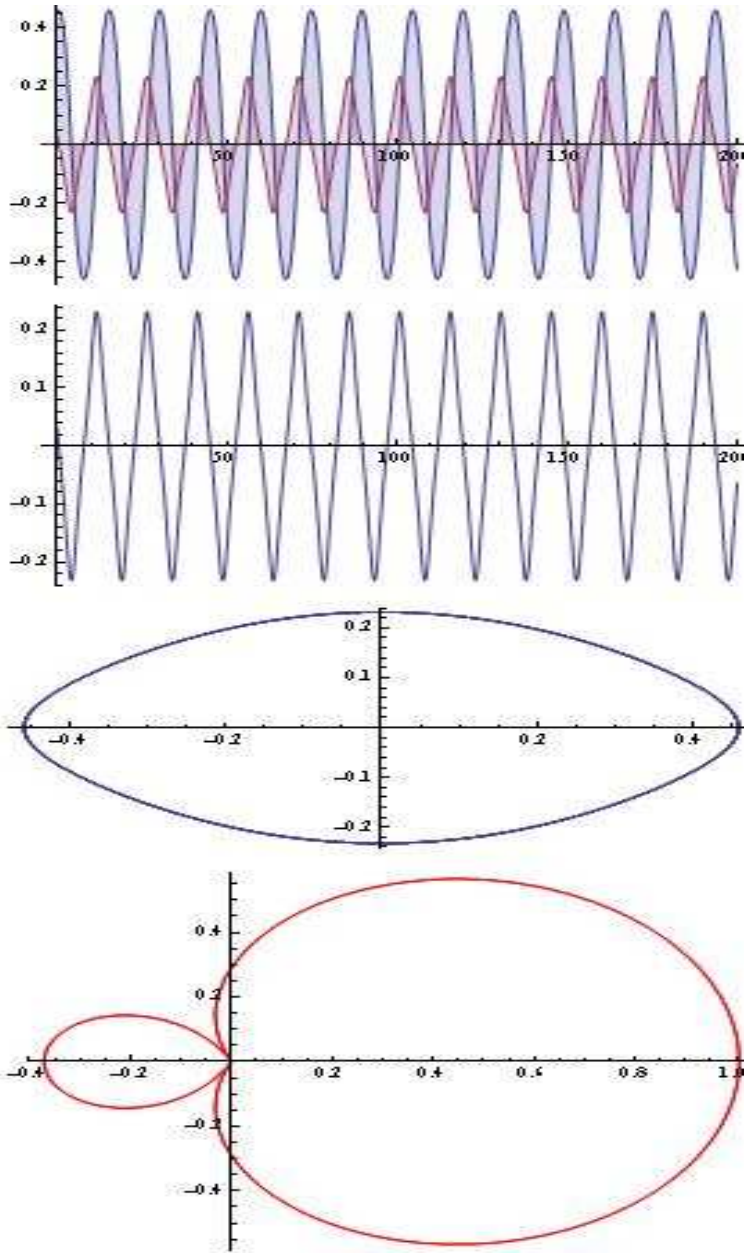


Figure 4: The simulations for $q = 0.5$, $x_0 = 1.35$, $y_0 = 0.9$, $b = 1.05$, $c = 1.08$, $\epsilon = 0.0001$: a) solutions of the system (2); b) y-component of the solution; c) the portrait; b) emitting chart (normalized).

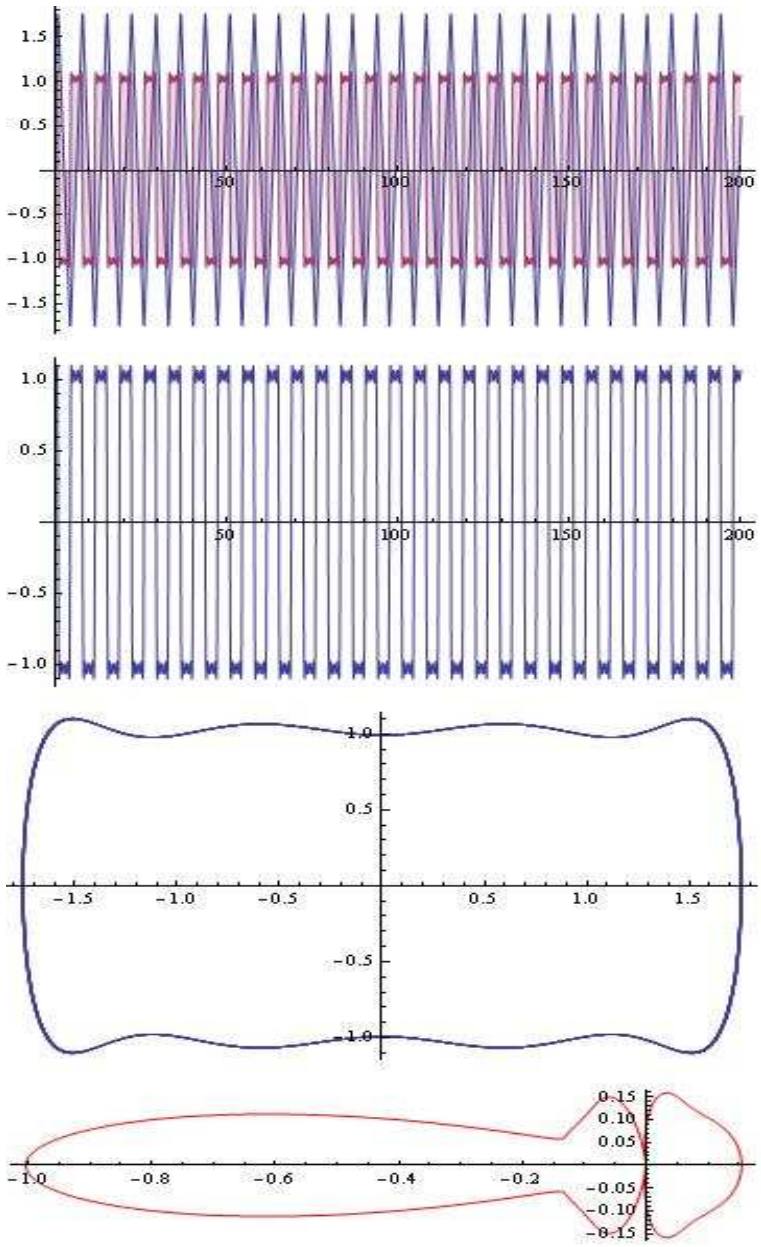


Figure 5: The simulations for $q = 0.5$, $s = 0.5$ $x_0 = 1.01$, $y_0 = 9$, $b = 0.99$, $c = 0.9$, $\epsilon = 0.0001$: a) solutions of the system (3); b) y-component of the solution; c) the portrait; b) emitting chart (normalized).

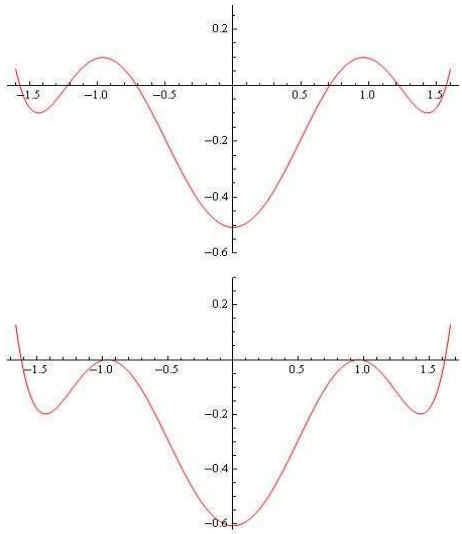


Figure 6: a) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$ and $\mu = 1.01442$ (three limit cycles); b) The Melnikov polynomial $M(r^2, 3)$ for $n = 7$ and $\mu = 1.21830512$ (simple limit cycle 1.06318 and limit cycle 0.959259 with multiplicity – two).

2.2. THE MODEL IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS

The Melnikov polynomial [8] for the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

is defined as

$$M(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 r^2 + \cdots + \left(\begin{array}{c} 2n+2 \\ n+1 \end{array} \right) \frac{a_{2n+1}}{2^{2n+2}} r^{2n}. \tag{5}$$

It is known [9]–[10] that the system for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial $M(r^2, n)$ has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

The case $n = 7$ *(*$s = q = 0.5$*).*

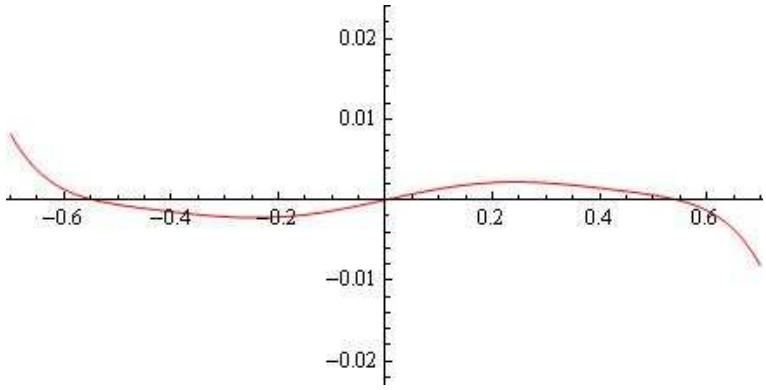


Figure 7: The function $F(x)$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(x^7 - 3.90022x^5 + 4.12927x^3 - \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{6}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Theorem. The Lienard–type system for $n = 7$, $(s = q = 0.5)$, and for all sufficiently small $\epsilon \neq 0$

a) for $\mu = 1.01442$ has three hyperbolic limit cycles 0.71176, 1.21853 and 1.57034.

b) for $\mu = 1.21830512$ has a simple limit cycle 1.61773 and limit cycle 0.959259 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 6) we have:

$$M(r^2, 3) = -\frac{\mu}{2} + \frac{359}{720}r^2 - \frac{223}{448}r^4 + \frac{35}{128}r^6. \tag{7}$$

Evidently, for example $\mu = 0.0735880318901$ we have a simple limit cycle and cycle with multiplicity – two.

3. CONCLUDING REMARKS

To minimize the noise in the antenna, other simulations related to the appropriate choice of the function $F(x)$ can be used.

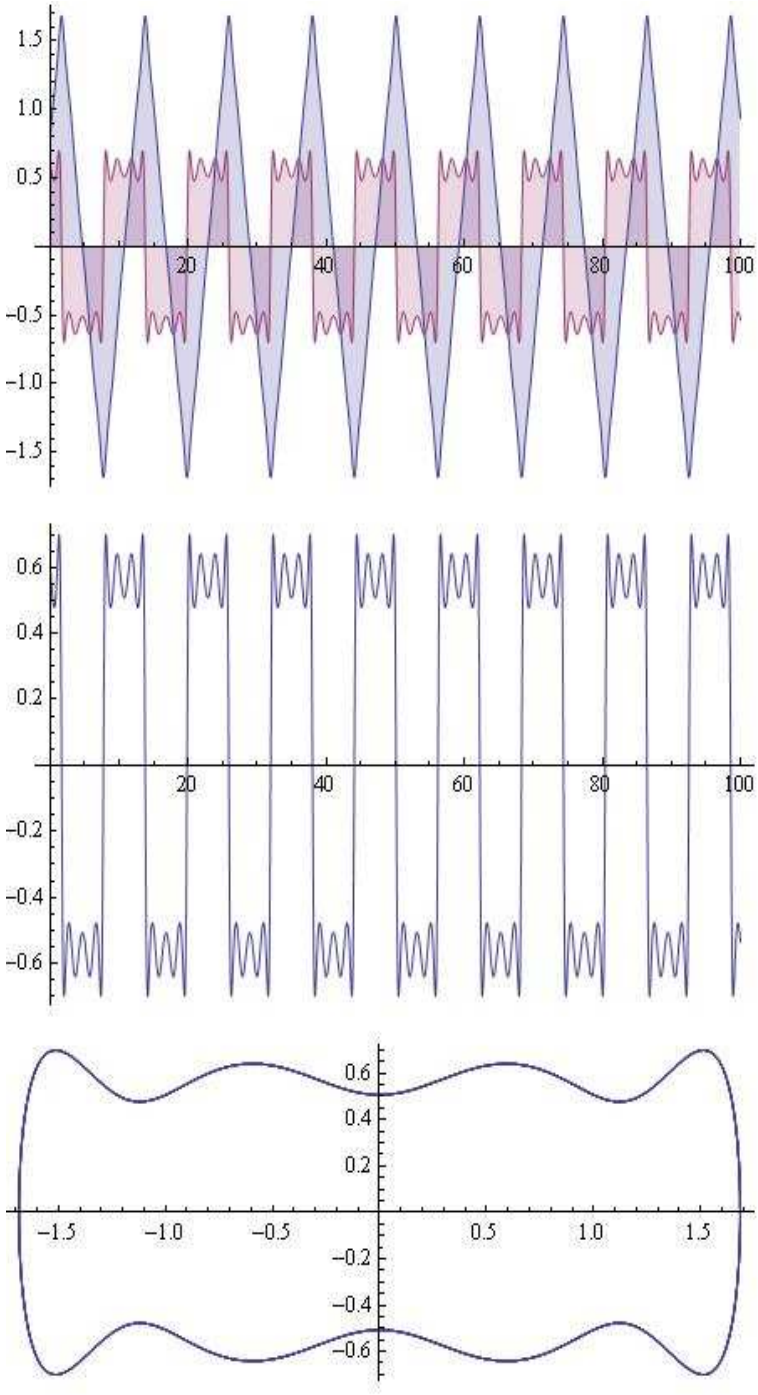


Figure 8: The simulations for $q = 0.5$, $s = 0.5$ $x_0 = 0.8$, $y_0 = 0.6$, $\epsilon = 0.0001$: a) solutions of the system; b) y-component of the solution; c) the portrait.

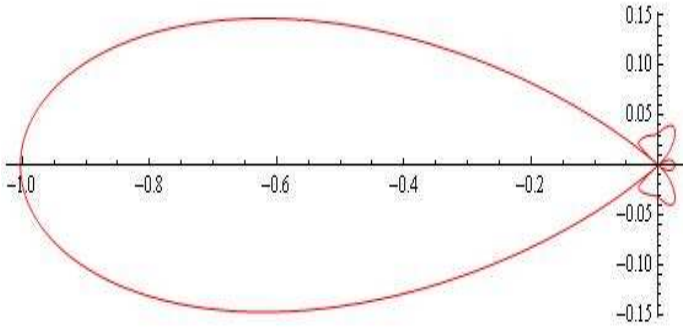


Figure 9: The normalized antenna factor ($b = 0.99$, $c = 0.79$).

So, for example, consider the system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -H_7(x,s,q) + \epsilon F(x) y \end{cases}$$

where the function $F(x)$ is of the form (see Fig. 7)

$$F(x) = \frac{6123}{414720}x - \frac{335855}{2903040}x^3 + \frac{794885}{2150400}x^5 - \frac{3527433}{7096320}x^7$$

The simulation on the system for $q = 0.5$, $s = 0.5$ $x_0 = 0.8$, $y_0 = 0.6$, $\epsilon = 0.0001$ is depicted on Fig. 8.

Let us define the normalized antenna factor as follows

$$X(\theta) = \frac{|y(b\cos\theta + c)|}{N}.$$

The emitting chart is depicted in Fig. 9.

Example 2. Let $F(x)$ is of the form (see Fig. 10)

$$F(x) = \frac{107}{20160}x - \frac{7351}{151200}x^3 + \frac{2131}{15120}x^5 - \frac{1}{8}x^7$$

The simulation on the system for $q = 0.5$, $s = 0.5$ $x_0 = 0.2$, $y_0 = 0.3$, $\epsilon = 0.0001$, $b = 0.99$, $c = 0.69$ is depicted on Fig. 11.

Let us mentioned that this idea could be applied successfully for approximation of the emitting chart of antenna pattern by using some orthogonal polynomials and associated polynomials considered in this paper.

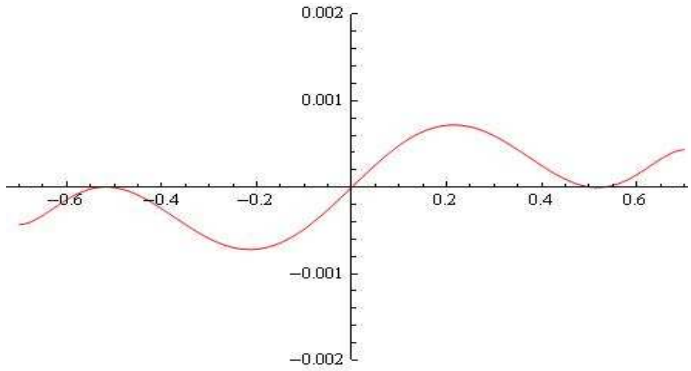


Figure 10: The function $F(x)$.

Acknowledgment

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0001-C01.

REFERENCES

[1] George E. Andrews, Richard Askey, Ranjan Roy, Special Functions, Cambridge Univ. Press, 1999.

[2] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials, Memoirs of the AMS 319, 1985.

[3] G. Gaspar, M. Rahman, Basic hypergeometric series, Second edition, Cambridge Univ. Press, 2003.

[4] R. Koekoek, R. Swarttouw, The Askey–scheme of hypergeometric orthogonal polynomials and its q–analogue

[5] J. Cigler, Continuous q–Hermite polynomials: an elementary approach, arXiv 1307.0357V2 [math HO] 24 Feb 2014.

[6] R. Koekoek, R. Askey, R. Swarttouw, Hypergeometric orthogonal polynomials and their q–analogues, Springer Monographs in Mathematics, Berlin, New York, Springer–Verlag, 2010.

[7] J. Cigler, J. Zang, A curious q–analogue of Hermit polynomials, *J. Comb. Th A*, 118 (2011), 9–20.

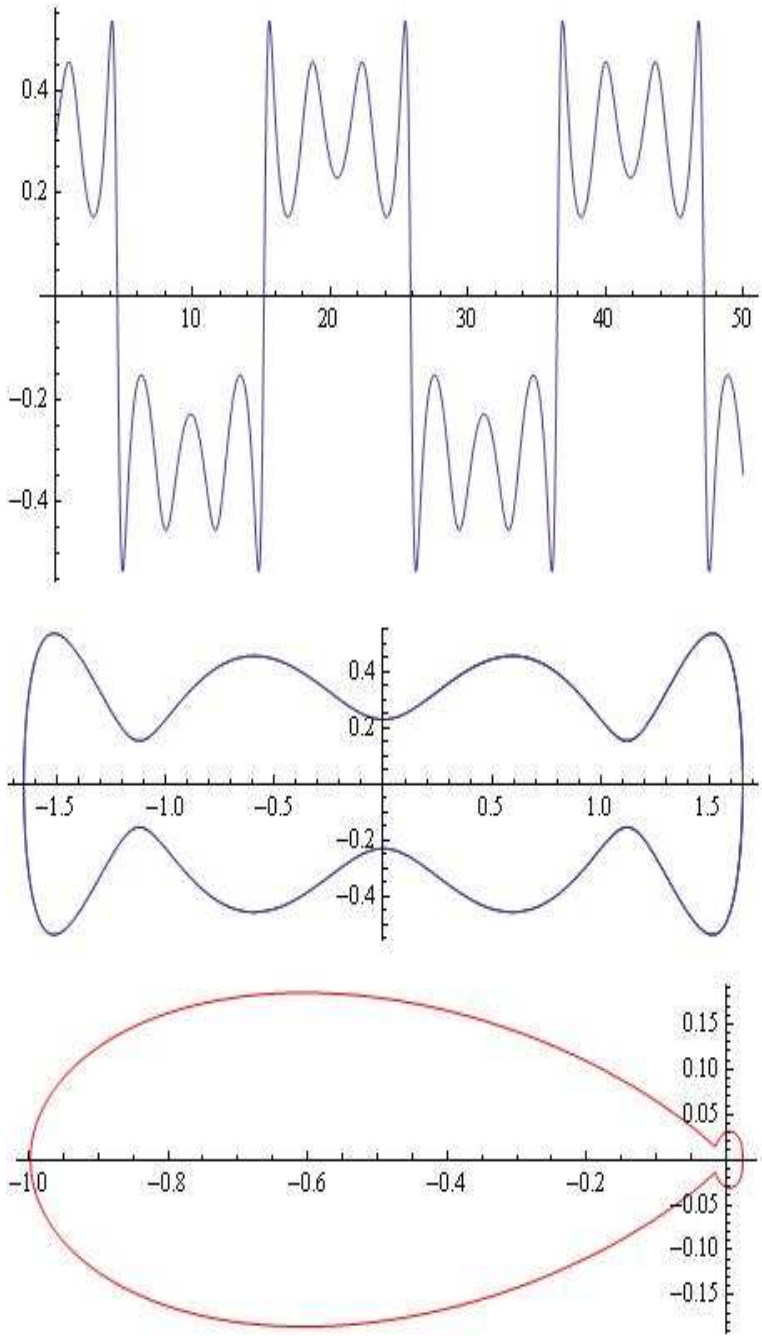


Figure 11: The simulations for $q = 0.5$, $s = 0.5$ $x_0 = 0.2$, $y_0 = 0.3$, $\epsilon = 0.0001$, $b = 0.99$, $c = 0.69$: a) y-component of the solution; b) the portrait; c) The normalized antenna factor.

[8] V. K. Melnikov, On the stability of a center for time–periodic perturbation, *Tr. Mosk. Mat. Obs.*, **12** (1963).

[9] T. Blows, L. Perko, *SIAM Rev.*, **36**, 341 (1994).

[10] L. Perko, *Differential Equations and Dynamical Systems*, Springer–Verlag, New York (1991).

[11] N. Kyurkchiev, A. Andreev, *Approximation and Antenna and Filters synthesis. Some Moduli in Programming Environment MATHEMATICA*, LAP LAMBERT Academic Publishing, Saarbrucken, 150 pp., (2014), ISBN: 978-3-659-53322-8.

[12] N. Kyurkchiev, Some Intrinsic Properties of Tadmor-Tanner Functions: Related Problems and Possible Applications, *Mathematics* (2020), **8**, 1963.

[13] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Simulations on the Lienard polynomial system with Dickson-type polynomial corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 31–44.

[14] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, A note on the extended Lienard system with Dickson polynomials of the third kind as corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 73–87.

[15] V. Kyurkchiev, N. Kyurkchiev, On an extended relaxation oscillator model: number of limit cycles, simulations. I, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[16] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A technique for simulating the dynamics of some extended relaxation oscillator models. II, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[17] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Another extended polynomial Lienard systems: simulations and applications. III, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No. 1 (2022), 55–65.

[18] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Investigations on some polynomial Lienard-type systems: number of limit cycles, simulations, *International Journal of Differential Equations and Applications*, **21**, No. 1 (2022), 117–126.

[19] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On some extended oscillator models: a technique for simulating and studying their dynamics*, Plovdiv, Plovdiv University Press (2022); ISBN 978-619-7663-13-6.

[20] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms* 15, 12, 2022, ISSN:1999-4893

[21] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Lienard System with First Kind Chebyshev's Polynomial–correction in the Light of Melnikov's Approach. Simulations and Possible Applications, International Scientific Conference IMEA'2022, 23-25 November 2022, Pamporovo, Bulgaria, ISBN: 978-619-7663-33-4, Plovdiv University Press, 105–112.

[22] E. Angelova, V. Arnaudova, T. Terzieva, A. Malinova, Investigations on a Differential System with Correction of Zernike-type Radial Polynomials. Simulations, International Scientific Conference IMEA'2022, 23-25 November 2022, Pamporovo, Bulgaria, ISBN: 978-619-7663-33-4, Plovdiv University Press, 87–92.

[23] N. Kyurkchiev, The effects on the dynamics of Lienard equation with Morse–type corrections: level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022).

[24] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, Legendre polynomials as "correction factors" in the Lienard differential system. Simulations, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[25] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, A. Golev, Associated Legendre polynomials as corrections in the classical Lienard system, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[26] A. Golev, V. Arnaudova, Lienard system with "correcting factors" of the type of interpolating polynomials of some basic functions, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No 1 (2022), 67–80.

[27] A. Golev, V. Arnaudova, Higher-Kind Chebyshev Polynomials and Their Applications to Dynamic Model Studies, *International Journal of Differential Equations and Applications*, 21, 2 (2022), 101–108.

[28] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Gegenbauer polynomials as correction in the Lienard planar system: Melnikov's approach, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 45–57.

[29] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On the Lienard System with Some "Corrections of Polynomial-type": Number of Limit Cycles, Simulations and Possible Applications. Part II.*, Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-41-9.

[30] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Dynamics of the Lienard Polynomial System Using Dickson Polynomials of the (M + 1)-th Kind. The Level Curves, *International Journal of Differential Equations and Applications*, 21, 2, 2022, 109–120.

[31] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, (2023). (accepted)

[32] Vasileva, M., Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Associated Hermite polynomials. Some applications, *International Journal of Differential Equations and Applications*, **22**, No. 1 (2023), 1–17.

[33] A. Golev, V. Arnaudova, Associated Gegenbauer polynomials. Applications, *International Journal of Differential Equations and Applications*, **22**, No. 1 (2023), 19–28.

[34] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Chebyshev polynomials of the fifth kind and their application to study the dynamics of Lienard-type equations, *Communications in Applied Analysis*, 26, 1, 2022, 111–123.

[35] M. Vasileva, A. Iliev, A. Rahnev, N. Kyurkchiev, Applications of Some Associated Orthogonal Polynomials, *Communications in Applied Analysis*, **27**, No. 1 (2023), 27–39.

[36] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A Look at the Associated Lommel Polynomials. Applications, *Communications in Applied Analysis*, **27**, No. 1 (2023), 41–52.

[37] A. Golev, V. Arnaudova, One possible application of the q–Lommel polynomials associated with the Jackson q-Bessel function, *International Electronic Journal of Pure and Applied Mathematics* (2023).