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DYNAMICS AND SIMULATIONS OF SOME GENERALIZED RAYLEIGH SYSTEMS (WEB PLATFORM UPGRADE)

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ABSTRACT: In this article we demonstrate some specialized modules for investigating the dynamics of some generalized Rayleigh systems, an integral part of a planned much more general Web–based application for scientific computing. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Rayleigh-Lienard differential system, generalized Rayleigh system, Duffing oscillator, emitting chart.

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1. INTRODUCTION

In [3] some investigations and simulations on the planar Rayleigh–Lienard system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon F(x,y)y \end{cases} \tag{1}$$

are given.

Here $0 \leq \epsilon < 1$; $F(x,y)$ and $Poly(x)$ are specially chosen polynomials.

For other results see [1], [4]–[9].

Define the normalized antenna factor

$$\frac{y(b\cos\theta + c)}{m}$$

where θ is the azimuthal angle and c is the phase difference. We note that the use of $y(\theta)$ - the solution component of the corresponding Rayleigh–Lienard differential system as an antenna factor is very complicated (see for more details [16], [14]).

A natural extension of the planned much more general Web–based application for scientific computing involves further consideration and simulations on generalized Rayleigh systems.

In this paper we demonstrate some specialized modules for investigating the dynamics of some generalized Rayleigh systems, an integral part of a planned much more general Web–based platform for scientific computing.

A new extended oscillator is investigated in Section 2.2.

Our considerations on the Melnikov homoclinic integral are summarized in Appendix 1.

A general Duffing oscillator in the light of Gavrilov–Gargouri–Hamed considerations is discussed in Section 2.3.

The dynamics of the planar systems with unbounded oscillation is followed in Section 2.4.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are also given.

2. MAIN RESULTS. SIMULATIONS

2.1. A LOOK AT THE PLANAR GENERALIZED RAYLEIGH SYSTEM

Consider a class of planar generalized Rayleigh system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + \epsilon(1 - y^n)y \end{cases} \tag{2}$$

where $0 < \epsilon < 1$.

For some details see [12].

I. For given $\epsilon = 0.001$, $b = 0.9$, $c = 0.79$ and $n = 30$ the simulation on the system (2) with $x_0 = 0.3$; $y_0 = 0.3$ is depicted on Fig. 1.

I.1 For given $\epsilon = -0.1$, $b = 0.4$, $c = 0.25$ and $n = 30$ the simulation on the system (2) with $x_0 = 0.3$; $y_0 = 0.3$ is depicted on Fig. 2.

Remark. It is easy to prove the existence of a unique limit cycle of the differential system (2) where the parameter $\epsilon \neq 0$.

2.2. A LOOK AT THE NEW EXTENDED OSCILLATOR

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Consider the following planar differential system:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon \left(1 + \displaystyle\sum_{i=1}^{n} (-1)^n a_i y^i\right) y \end{cases} \tag{3}$$

where $0 < \epsilon < 1$.

II. For given $n = 9$, $\epsilon = 0.5$, $b = 0.12$, $c = 0.73$;
$Poly(x) = x - \frac{1}{3}x^3$ (Van der Poll's correction) and

$$a_1 = 0.1111; a_2 = 0.1; a_3 = 0.1; a_4 = 0.001; a_5 = 10$$

$$a_6 = 1; a_7 = 0.2; a_8 = 1.1; a_9 = 2$$

the simulation on the system (3) with $x_0 = 0.5$; $y_0 = 0.4$ is depicted on Fig. 3.

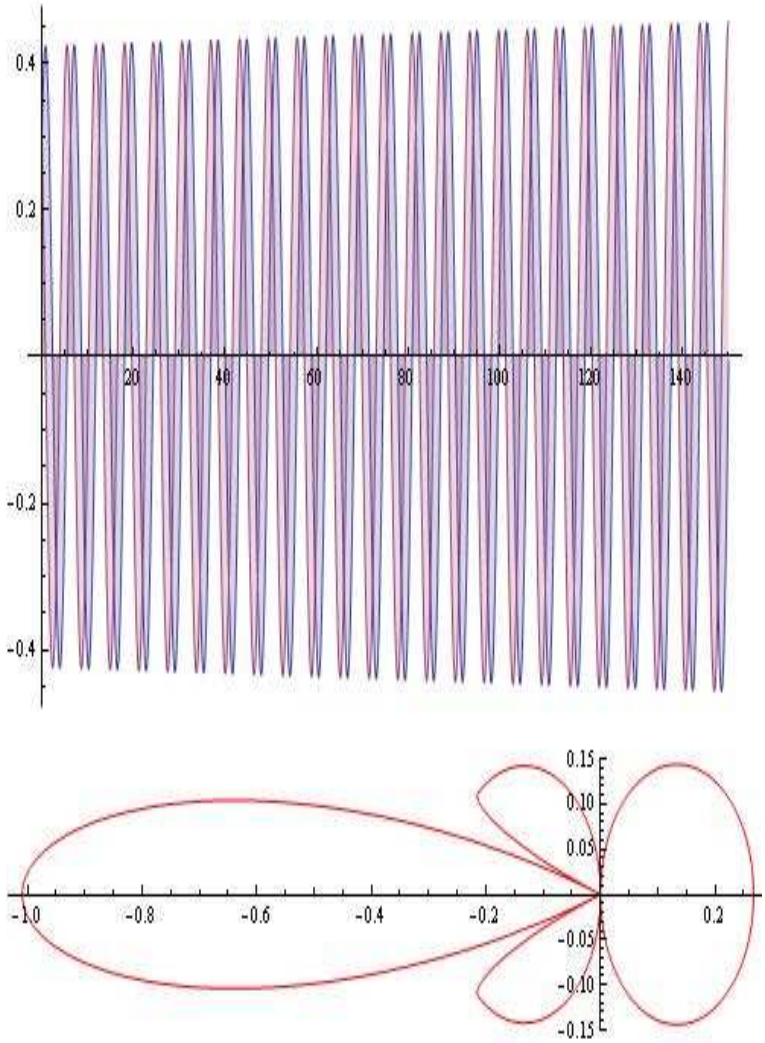


Figure 1: a) The solutions of differential system; b) radial diagram (example I.)

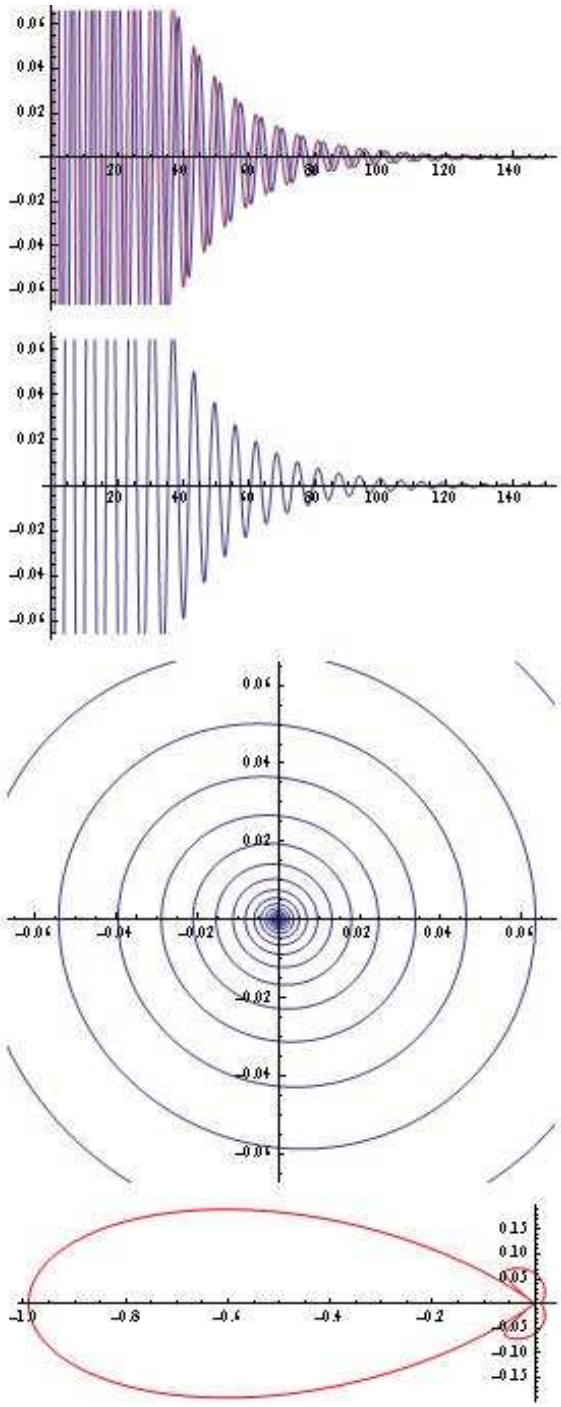


Figure 2: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example I.1)

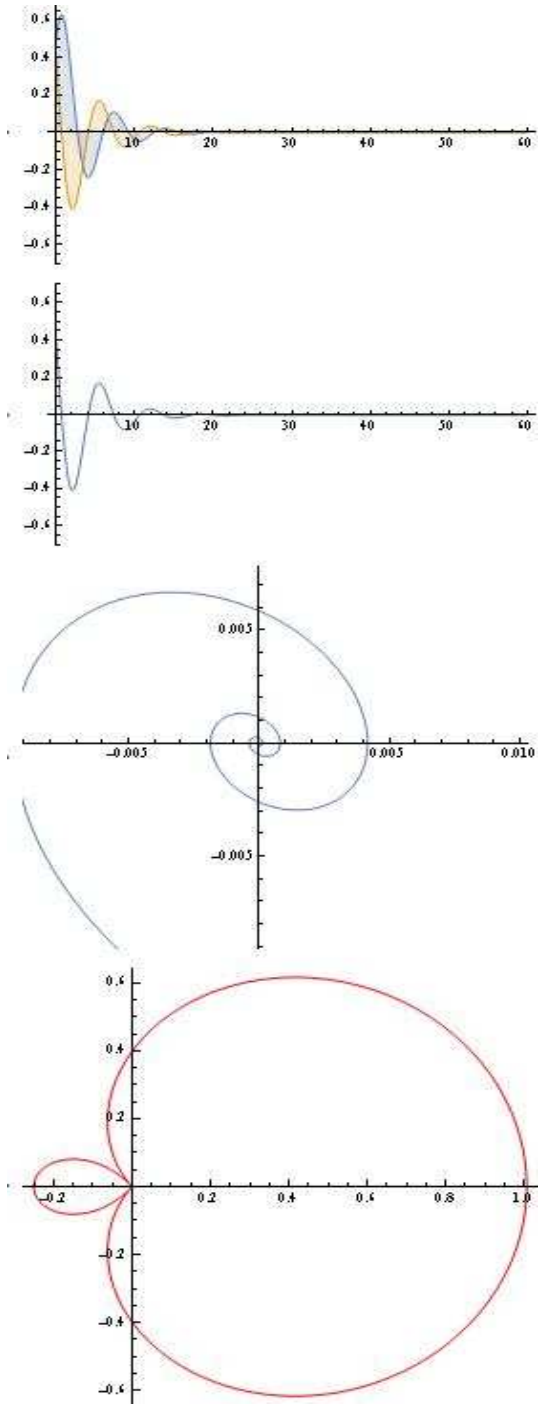


Figure 3: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example II.)

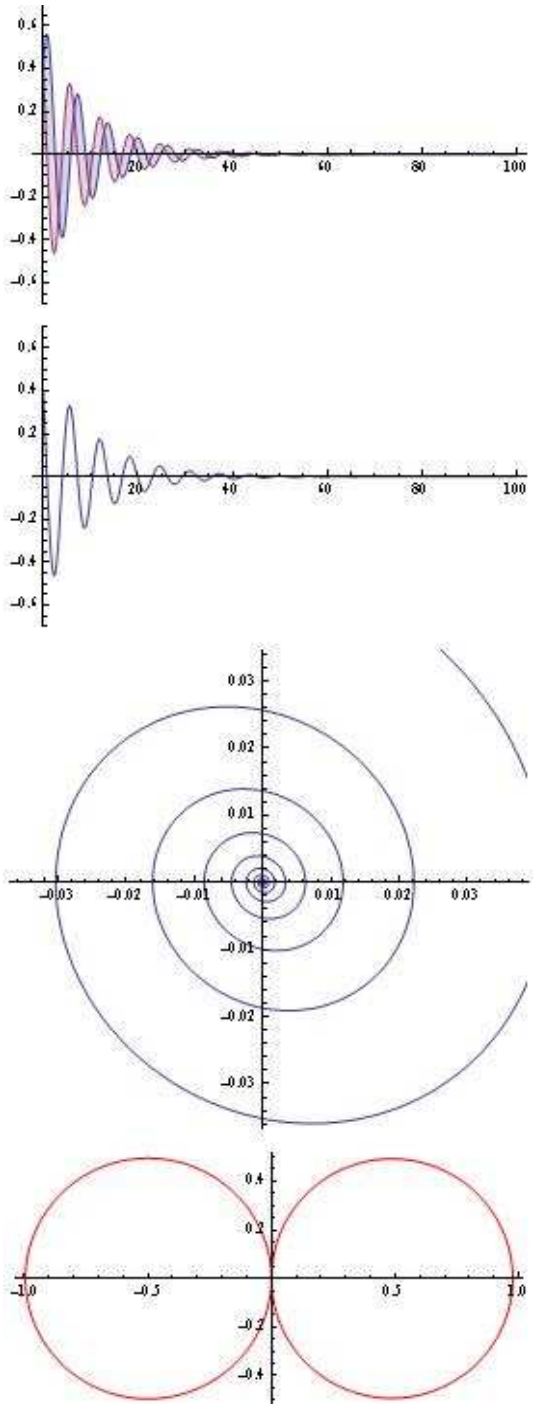


Figure 4: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (dipole) - example II.1

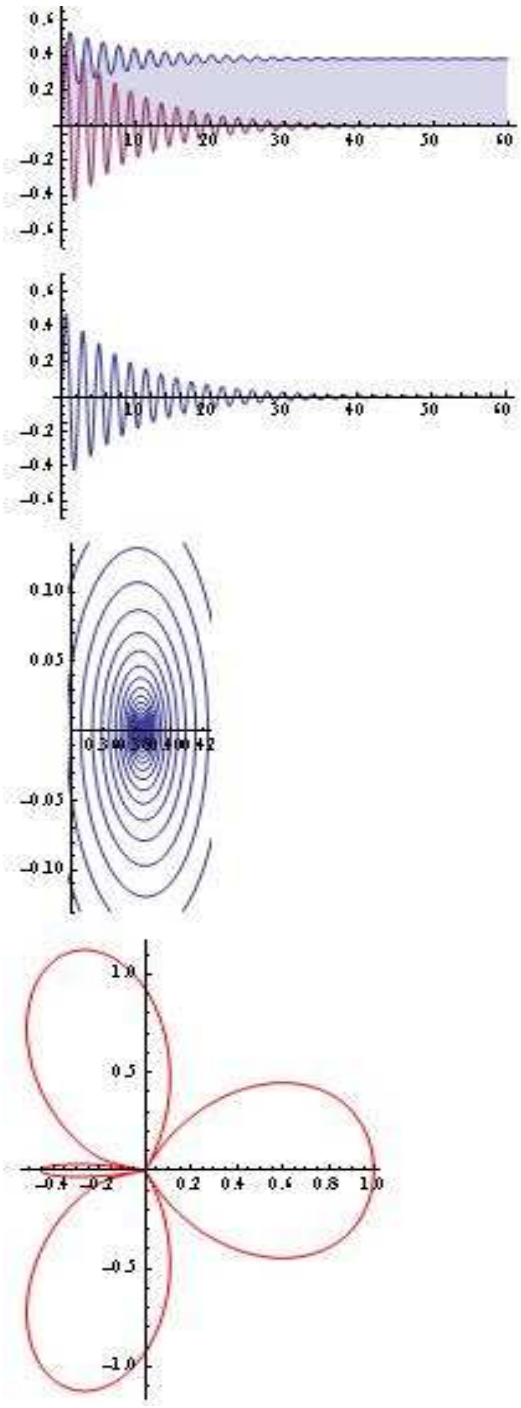


Figure 5: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example II.2)

Remark. Our considerations on the Melnikov homoclinic integral are summarized in Appendix 1.

II.1 For given $n = 9$, $\epsilon = 0.2$, $b = 0.16$, $c = 0.797$ and

$$a_1 = 0.12; a_2 = 0.2; a_3 = 0.2; a_4 = 0.1; a_5 = 0.5$$

$$a_6 = 0.9; a_7 = 0.4; a_8 = 0.8; a_9 = 1.2$$

the simulation on the system (3) with $x_0 = 0.4$; $y_0 = 0.4$ is depicted on Fig. 4.

II.2 For given $n = 9$, $\epsilon = 0.2$, $b = 0.79$, $c = 0.74$

$Poly(x) = x^7 - 192x^5 + 80x^3 - 8x$ (Chebyshev polynomial of second kind as "correction") and

$$a_1 = 0.05; a_2 = 0.3; a_3 = 0.6; a_4 = 0.07; a_5 = 0.65$$

$$a_6 = 0.95; a_7 = 0.25; a_8 = 0.5; a_9 = 1.8$$

the simulation on the system (3) with $x_0 = 0.2$; $y_0 = 0.1$ is depicted on Fig. 5.

2.3. A LOOK AT THE MORE GENERAL DUFFING OSCILLATOR

Gavrilov, Gargouri and Hamed [13] find an upper bound for the number of limit cycles bifurcating from the 8–loop of the *Duffing* oscillator

$$x'' = x - x^3 \tag{4}$$

under the special cubic perturbation

$$x'' = x - x^3 + a_1 y + a_2 x^2 + a_3 xy + a_4 x^2 y. \tag{5}$$

The perturbed Duffing oscillator is defined as

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - x^3 + a_1 y + a_4 x^2 y \end{cases} \tag{6}$$

where a_i are small parameters.

The authors [13] define the following more general deformation

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - x^3 + a_1 y + a_2 x^2 + a_3 xy + a_4 x^2 y \end{cases} \tag{7}$$

where a_i are small parameters.

Obviously, the differential system (7) falls into our proposed much more general classification scheme of the form:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon G(x,y)y \end{cases} \tag{8}$$

where $0 \leq \epsilon < 1$; $G(x,y)$ is of the type

$G(x,y) = a_1 + a_2\frac{x^2}{y} + a_3 x + a_4 x^2$ (a_j are real parameters) and $Poly(x)$ are specially chosen polynomials as "corrections" in the differential system.

III. Let $Poly(x) = 2x^3 - x$. For given $\epsilon = 0.0001$, $b = 1.9$; $c = 0.79$ the simulations on the system (8) for

1. $x_0 = 0.8$; $y_0 = 0.6$ and

$$a_1 = 7.9; a_2 = -1.5; a_3 = 23.8; a_4 = 8.6$$

2. $x_0 = 0.6$; $y_0 = 0.4$ and

$$a_1 = 10; a_2 = -10; a_3 = 20; a_4 = 7$$

are depicted on Fig. 6–Fig. 7.

2.4. A LOOK AT THE PLANAR SYSTEMS WITH UNBOUNDED OSCILLATION

It is known (see [2]) that the system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= \left(\frac{y}{1+y^2} - x\right)(1+y) \end{cases} \tag{9}$$

has an unbounded oscillation.

The system (9) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - \epsilon\left(\frac{1}{1+y^2} + \frac{y}{1+y^2} + x\right)y \end{cases} \tag{10}$$

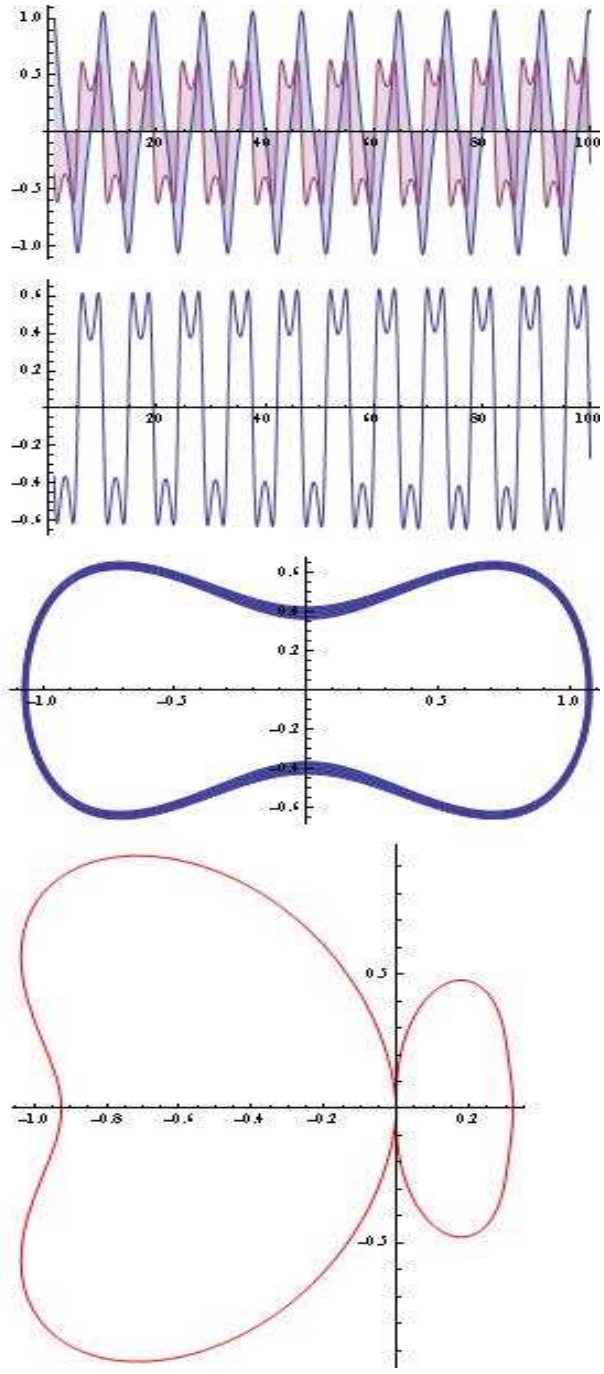


Figure 6: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example III.1).

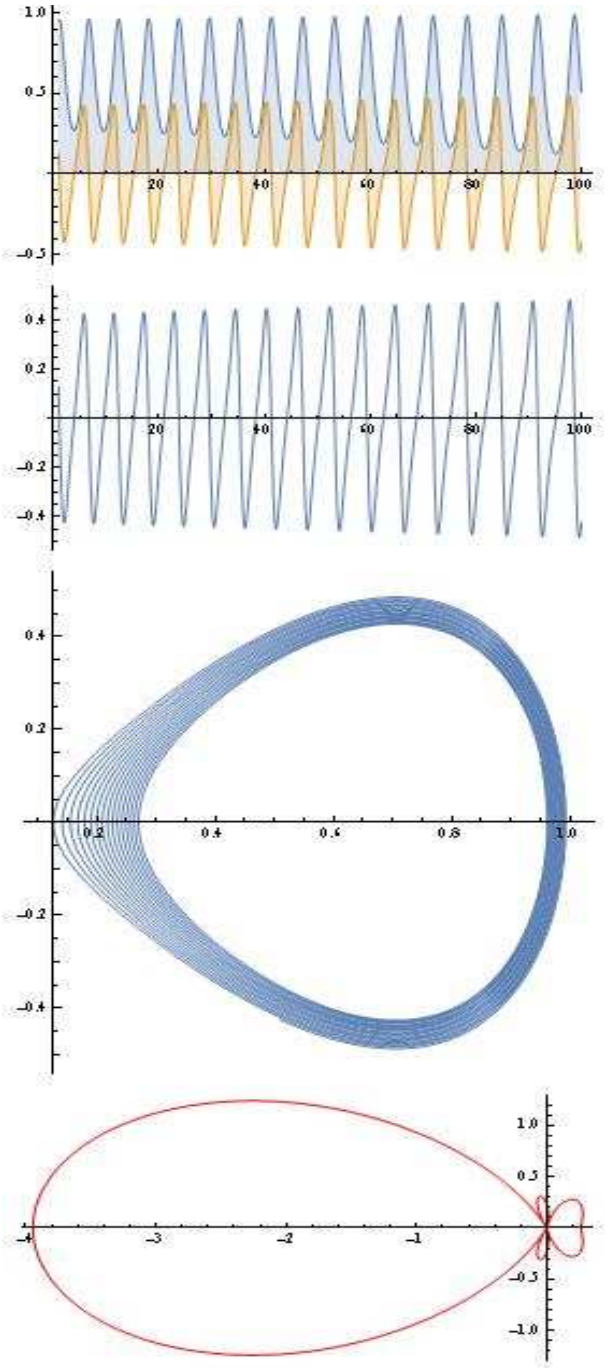


Figure 7: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example III.2).

Obviously, the differential system (10) falls into our proposed much more general classification scheme of the form:

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon G_1(x,y)y \end{cases} \tag{11}$$

where $0 \leq \epsilon < 1$; $Poly(x) = -x$, $G_1(x,y)$ is of the type
$G_1(x,y) = \frac{1}{1+y^2} + \frac{y}{1+y^2} + x$.

IV. For given $\epsilon = 0.0001$, $b = 0.95$; $c = 0.79$ the simulations on the system (11) for $x_0 = 0.7$; $y_0 = 0.5$ are depicted on Fig. 8.

2.5. ANOTHER MODELS

Consider the planar system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -Poly(x) + \epsilon G_2(x,y)y \end{cases} \tag{12}$$

where

$$Poly(x) = x - \epsilon(a_1 + a_2 x + a_3 x^3);$$
$$G_2(x,y) = x + 3x^2 - y^2.$$

V. For given $\epsilon = 0.01$, $b = 0.92$; $c = 0.789$, $a_1 = 13$, $a_2 = 23$, $a_3 = 14$ the simulations on the system (12) for $x_0 = 0.7$; $y_0 = 0.5$ are depicted on Fig. 9.

3. CONCLUDING REMARKS

In this paper we demonstrate some specialized modules for investigating the dynamics of some generalized Rayleigh systems, an integral part of a planned much more general Web–based application for scientific computing.

We presented only a small part of the platform's capabilities. We will be grateful to all colleagues who, with their critical remarks, will contribute to its significant improvement.

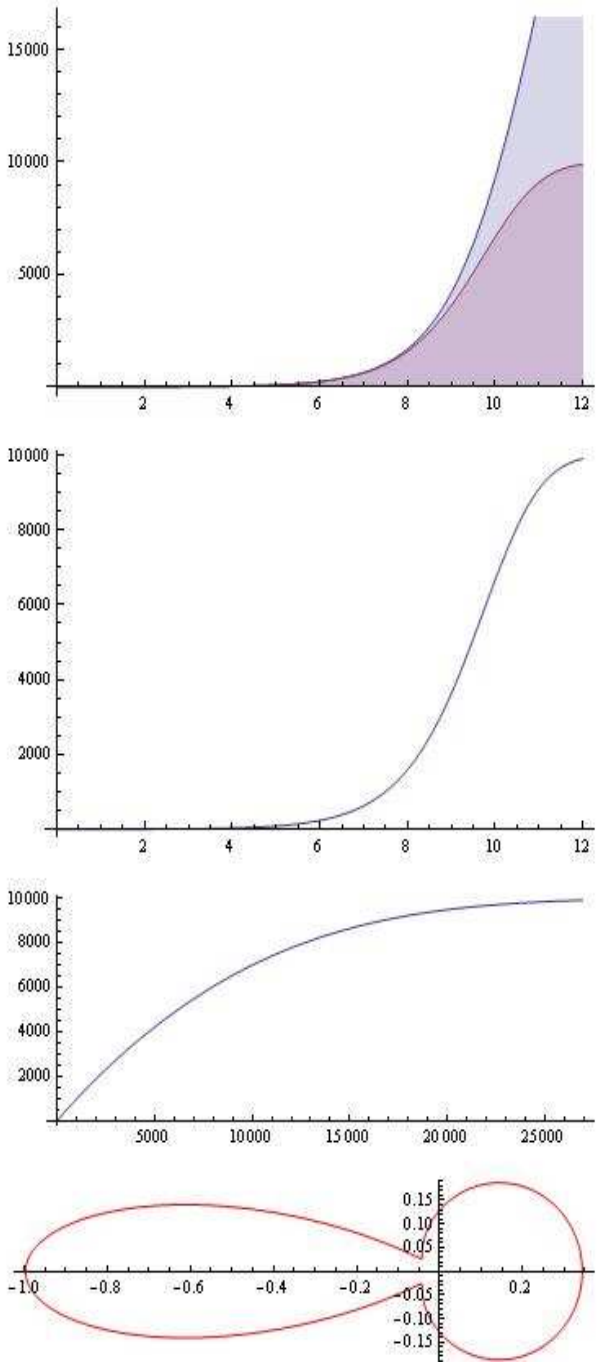


Figure 8: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example IV).

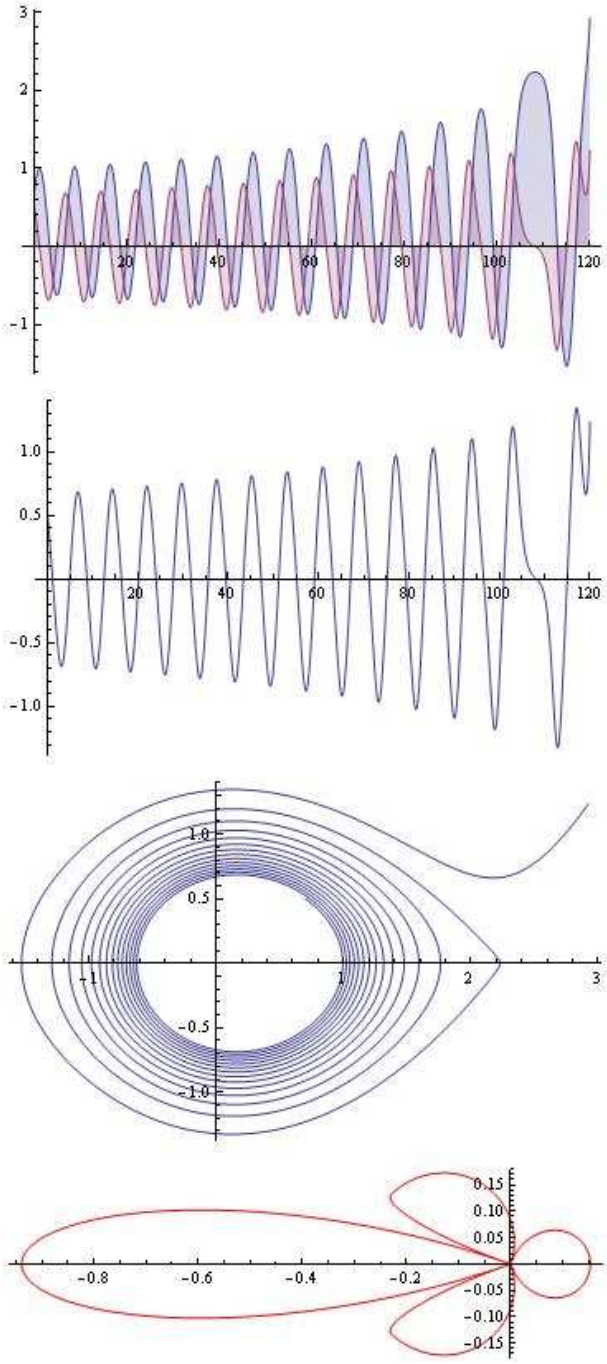


Figure 9: a) The solutions of differential system; b) y-component of the solution; c) Phase portrait; d) radial diagram (example V).

We use a algorithm for control and visualization of the "diagram factor" (with a possible user–set value of the lateral radiation).

In [15] the authors study the global dynamics of a van der Pol-Duffing oscillator with indefinite degree.

Our Web–based application also provides specialized modules for studying the dynamics of the above-mentioned class of oscillators, related to the study of:

- the local bifurcation diagram and corresponding local phase portraits of planar system;

- the bifurcation diagram and corresponding global phase portraits in the Poincare disc of planar system

and other.

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4. APPENDIX 1.

The case **II.**. For example the system of the type (3) ($n = 3$ and $x - \frac{1}{3}x^3$ is the Van der Poll's "correction")

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= x - \frac{1}{3}x^3 + \epsilon\left(1 - a_1 y + a_2 y^2 - a_3 y^3\right) y \end{cases}$$

has the following Hamiltonian ($\epsilon = 0$)

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{12}x^4.$$

The homoclinic orbit is given by (see Fig.10)

$$x_0(t) = \pm\sqrt{6}sech\, t$$
$$y_0(t) = \mp\sqrt{6}sech\, t\, tanh\, t.$$

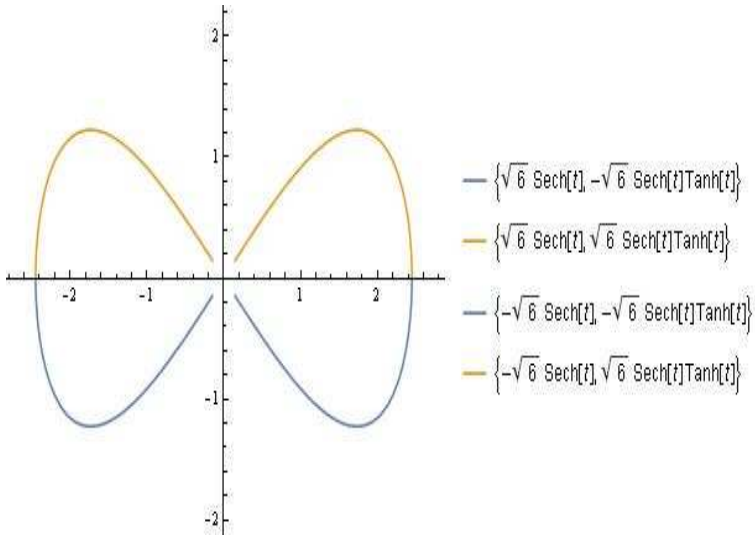


Figure 10: The homoclinic orbit.

The Melnikov homoclinic integral can be computed explicitly

$$M(a_1;a_2;a_3) =$$

$$= \int_{-\infty}^{\infty} y_0(t)\left(1 - a_1 y_0(t) + a_2 y_0^2(t) - a_3 y_0^3(t)\right) y_0(t)\, dt =$$

$$= \int_{-\infty}^{\infty} (-\sqrt{6} sech\, t\, tanh\, t)^2 \left(1 - a_1(-\sqrt{6} sech\, t\, tanh\, t) + a_2(-\sqrt{6} sech\, t\, tanh\, t)^2 - \right.$$

$$\left. - a_3(-\sqrt{6} sech\, t\, tanh\, t)^3\right)\, dt$$

From a numerical point of view, the task of finding a root of $M(a_1;a_2;a_3)$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions (of a physical and practical nature).

We have

$$M(a_1;a_2;a_3) = 4 + \frac{144}{35} a_2.$$

Thus we prove the following

Proposition. If

$$a_2 = -\frac{35}{36}$$

then the Melnikov function $M(a_1;a_2;a_3)$ has root.

Remark. The study of corresponding critical levels of $H(x,y) = \frac{1}{2}y^2 - P(x)$ is very complicated.

In this regard, we recommend the study by Gavrilov and Iliev [19].

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