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FUZZY SUMUDU TRANSFORM TO SOLVE CONVOLUTION TYPE VOLTERRA FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, we discussed the application of fuzzy Sumudu transform (FST) to solve Volterra fuzzy integro-differential equation (VFIDE) of first kind with convolution kernel under Hukuhara differentiability. It is shown that FST is a simple and reliable approach for solving such equations analytically. Finally, the method is illustrated with example to show the efficiently of the proposed method.

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1. INTRODUCTION

In the recent years, the area of fuzzy integro-differential equations (FIDE) has developed a lot and plays a key role in the field of engineering. Furthermore, FIDEs in a fuzzy setting are a natural way to model the ambiguity of dynamic systems. Consequently, different scientific fields, such as physics, geography, medicine, and biology, pay much importance to the solution of different FIDE. Solutions to these equations can be utilized in different engineering problems. Seikkala first defined fuzzy derivatives, while the concept of integration of fuzzy functions was first introduced by Dubois and Prade. One of the most important field of the fuzzy theory is the fuzzy differential equations [3, 4, 11], fuzzy integral equations [12] and fuzzy integro-differential equations [5, 6, 7, 9, 14].

In the 1990's Watugala [20] has introduced a new integral transform called the Sumudu transform. In [1, 5] is proposed the idea of the fuzzy method of transformation of Sumudu to solve fuzzy partial differential and integro-differential equations. The technique of the fuzzy Sumudu transform method for solving a fuzzy convolution Volterra integral equations and the fuzzy integro-differential equation was developed in [2] and [13]. Sumudu transform along with broad applications has been utilized in the area of system engineering and applied physics.

In this paper, we use of fuzzy Sumudu transform for solving the Volterra fuzzy integro-differential equation of first kind with convolution kernel

$$\int_0^x k_1(x-s) \odot w(s))ds \oplus \int_0^x k_2(x-s) \odot w^{(n)}(s))ds = g(x), \tag{1.1}$$

with the initial conditions

$$w^{(i)}(0) = b_i, \quad i = 0, 1, 2, ..., n-1, \tag{1.2}$$

where $k_2(x-s) \neq 0$, $k_1,\ k_2 : [a,b] \times [a,b] \to \mathbb{R}$, are continuous functions on E^1 and $g,\ u : [a,b] \to E^1$ are continuous fuzzy-number valued functions and $b_i,\ (i = 0, 1, ..., n-1)$ are constants. The set E^1 is the set of all fuzzy numbers.

By using fuzzy Sumudu transform method, we directly convert Volterra fuzzy integro-differential equation of first kind with convolution kernel into an algebraic equation. Solving this algebraic equation and applying fuzzy inverse Sumudu transform we obtain the exact solution. This method is illustrated by giving of example.

2. PRELIMINARIES

In this section, we give some basics definitions and theorems for fuzzy number, fuzzy-valued function and derivative of fuzzy-valued function.

Definition 2.1. *[8] A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties*

1. *u is upper semi-continuous on $\mathbb{R}$,*
2. *$u(x) = 0$ outside of some interval $[c,d]$,*
3. *there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$,*
4. *u is fuzzy convex set i. e. that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.*

The set of all fuzzy numbers is denoted by E^1. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi_{[a]}$ and therefore $\mathbb{R} \subset E^1$.

Definition 2.2. *[10] Let $u \in E^1$ and $r \in (0,1]$. The r-level set of u is the crisp set $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$, where $[u]^r$ denotes r-level set of fuzzy number u.*

It can be concluded that any r-level set is bounded and closed interval and denoted by $[\underline{u}(r), \overline{u}(r)]$ for all $r \in [0,1]$, where the functions $\underline{u}$, $\overline{u} : [0,1] \to \mathbb{R}$ are the lower and upper bound of $[u]^r$, respectively.

Definition 2.3. *[10] A fuzzy number in parametric form is given as an order pair of the form $u = (\underline{u}(r), \overline{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions*

1. *$\underline{u}(r)$ is a bounded left continuous monotonic increasing function in $[0,1]$,*

2. *$\overline{u}(r)$ is a bounded left continuous monotonic decreasing function in $[0,1]$,*

3. *$\underline{u}(r) \leq \overline{u}(r)$.*

For arbitrary fuzzy number $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and an arbitrary crisp number $k \in \mathbb{R}$ the addition and the scalar multiplication are defined by $[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$ and

$$[k \odot u]^r = k.[u]^r = \begin{cases} [k\underline{u}(r), k\overline{u}(r)], & k \geq 0 \\ [k\overline{u}(r), k\underline{u}(r)], & k < 0. \end{cases}$$

The neutral element respect to $\oplus$ in E^1, denoted by $\tilde{0} = \chi_{[0]}$. The algebraic properties of addition and scalar multiplication of fuzzy numbers are given in [18].

Definition 2.4. *[15] Let $x, y \in E^1$ and exists $z \in E^1$, such that $x = y \oplus z$. Then z is called the H-difference of x and y and is given by $x \ominus y$.*

We use the Hausdorff metric as a distance between fuzzy numbers.

Definition 2.5. *[8] For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$ the quantity*

$$d(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$$

is the distance between u, v.

The metric d is a complete metric space in E^1.

For any fuzzy-number-valued function $f : I \subset \mathbb{R} \to E^1$ we can define the functions $\underline{f}(.,r),\ \overline{f}(.,r) : I \subset \mathbb{R} \to \mathbb{R}$, by $\underline{f}(t,r) = \underline{f(t,r)}$ $\overline{f}(t,r) = \overline{f(t,r)}$ for each $t \in I$, for each $r \in [0,1]$. These functions are called the left and right $r-$level functions of f.

Theorem 2.1. *[19] Let for all $r \in [0,1]$ the functions $\underline{f}(x,r)$ and $\overline{f}(x,r)$ are Riemann-integrable on $[0,b]$ for every $b \geq 0$ and there are constants $\underline{M}(r) > 0$ and $\overline{M}(r) > 0$, such that*

$$\int_0^b |\underline{f}(x,r)|dx \leq \underline{M}(r), \quad \int_0^b |\overline{f}(x,r)|dx \leq \overline{M}(r), \quad for\ every\ b \geq a.$$

Then the function $f(x)$ is improper fuzzy Riemann-integrable on $[0,\infty)$ and

$$(FR)\int_0^\infty f(x)dx = \left(\int_0^\infty \underline{f}(x,r)dxdy, \int_0^\infty \overline{f}(x,r)dx\right).$$

Definition 2.6. *[15] Let $x,\ y \in E^1$, if there exists $z \in E^1$ such that $x = y \oplus z$ then z is called the H-difference of x and y and it is given by $x \ominus y$.*

For fuzzy valued functions $w = w(x)$ we define the H-derivatives as given in [16, 17].

Definition 2.7. *Let $w : (a,b) \to E^1$, then w is said to be strongly generalized H-differentiable function at $x_0 \in (a,b)$, if there exists an element $w'(x_0) \in E^1$ such that*

1. *for all $h > 0$ sufficiently small the H-differences $w(x_0+h) \ominus w(x_0)$, $w(x_0) \ominus w(x_0+h)$, exist and the following limits hold*

$$\lim_{h\to 0} \frac{w(x_0+h) \ominus w(x_0)w}{h} = \lim_{h\to 0} \frac{w(x_0) \ominus w(x_0+h)}{h} = w'(x_0)$$

2. *for all $h > 0$ sufficiently small the H-differences $w(x_0) \ominus w(x_0+h)$, $w(x_0-h) \ominus w(x_0)$, exist and the following limits hold*

$$\lim_{h\to 0} \frac{w(x_0) \ominus w(x_0+h)}{-h} = \lim_{h\to 0} \frac{w(x_0-h) \ominus w(x_0)}{-h} = w'(x_0)$$

The first type of differentiability as in Definition 7 is referred as (i)-differentiable, while the second type as (ii)-differentiable.

Theorem 2.2. *[3] Let $w : (a,b) \to E^1$ be a continuous fuzzy-valued function and $w(x) = (\underline{w}(x,r), \overline{w}(x,r))$ for all $r \in [0,1]$. Then*

1. *if $w(x)$ is (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are differentiable and*

$$w'(x) = \left(\underline{w}'(x,r),\ \overline{w}'(x,r)\right). \tag{2.1}$$

2. *if $w(x)$ is (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are differentiable and*

$$w'(x) = \left(\overline{w}'(x,r),\ \ \underline{w}'(x,r)\right). \tag{2.2}$$

For fuzzy valued functions $w = w(x)$ we define the n-th order H-derivatives as given in [9].

Definition 2.8. *Let $w : (a,b) \to E^1$, then w is said to be strongly generalized H-differentiable function of the n-th order at $x_0 \in (a,b)$, if there exists an element $w^{(n)}(x_0) \in E^1$ such that*

1. *for all $h > 0$ sufficiently small the H-differences $w^{(n-1)}(x_0+h) \ominus w^{(n-1)}(x_0)$, $w^{(n-1)}(x_0) \ominus w^{(n-1)}(x_0+h)$, exist and the following limits hold*

$$\lim_{h\to 0} \tfrac{w^{(n-1)}(x_0+h)\ominus w^{(n-1)}(x_0)}{h} =$$
$$= \lim_{h\to 0} \tfrac{w^{(n-1)}(x_0)\ominus w^{(n-1)}(x_0+h)}{h} = w^{(n)}(x_0),$$

2. *for all $h > 0$ sufficiently small the H-differences $w^{(n-1)}(x_0) \ominus w^{(n-1)}(x_0+h)$, $w^{(n-1)}(x_0-h) \ominus w^{(n-1)}(x_0)$, exist and the following limits hold*

$$\lim_{h\to 0} \tfrac{w^{(n-1)}(x_0)\ominus w^{(n-1)}(x_0+h)}{-h} =$$
$$= \lim_{h\to 0} \tfrac{w^{(n-1)}(x_0-h)\ominus w^{(n-1)}(x_0)}{-h} = w^{(n)}(x_0).$$

Similar to the Theorem 2. we have the following results for n-th order strongly generalized H-differentiability of fuzzy-valued function.

Theorem 2.3. *[3] Let $w(x)$, $w'(x)$, ..., $w^{(n-1)}(x)$ be differentiable fuzzy-valued functions and $w(x) = (\underline{w}(x,r), \overline{w}(x,r))$ for all $r \in [0,1]$. Then*

1. *if $w(x)$ and $w'(x)$, ..., $w^{(n-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$w^{(n)}(x) = \left(\underline{w}^{(n)}(x,r),\ \ \overline{w}^{(n)}(x,r)\right).$$

2. *if $w(x)$ and $w'(x)$, ..., $w^{(n-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$w^{(n)}(x) = \left(\overline{w}^{(n)}(x,r),\ \ \underline{w}^{(n)}(x,r)\right) \quad if \ \ n \ is \ even,$$
$$w^{(n)}(x) = \left(\underline{w}^{(n)}(x,r),\ \ \overline{w}^{(n)}(x,r)\right) \quad if \ \ n \ is \ odd.$$

3. *if* $w(x)$ *is* (i)*-differentiable and* $w'(x)$, ..., $w^{(n-1)}(x)$ *are* (ii)*-differentiable, then* $\underline{w}(x,r)$ *and* $\overline{w}(x,r)$ *are H-differentiable of the* n*-th order and*

$$w^{(n)}(x) = \left(\overline{w}^{(n)}(x,r),\ \underline{w}^{(n)}(x,r)\right) \ \ if \ \ n \text{ is even},$$
$$w^{(n)}(x) = \left(\underline{w}^{(n)}(x,r),\ \overline{w}^{(n)}(x,r)\right) \ \ if \ \ n \text{ is odd}.$$

4. *if* $w(x)$ *is* (ii)*-differentiable and* $w'(x)$, ..., $w^{(n-1)}(x)$ *are* (i)*-differentiable, then* $\underline{w}(x,r)$ *and* $\overline{w}(x,r)$ *are H-differentiable of the* n*-th order and*
$$w^{(n)}(x) = \left(\overline{w}^{(n)}(x,r),\ \underline{w}^{(n)}(x,r)\right).$$

3. FUZZY SUMUDU TRANSFORM

Definition 3.1. *Let* $w : \mathbb{R} \to E^1$ *be a continuous fuzzy-valued function and the function* $e^{-x} \odot w(vx)$ *is improper fuzzy Riemann-integrable on* $[0, \infty)$*, then*

$$(FR)\int_0^\infty e^{-x} \odot w(vx)dx,$$

is called fuzzy Sumudu transform (FST) and is denote by

$$W(v) = S[w(x)] = (FR)\int_0^\infty e^{-x} \odot w(vx)dx, \tag{3.1}$$

for $v \in [-\sigma_1, \sigma_2]$*, where the variable* v *is used to factor the variable* x *in the argument of the fuzzy-valued function and* $\sigma_1,\ \sigma_2 > 0$.

The parametric form of FST is follows

$$S[w(x)] = (s[\underline{w}(x,r)], s[\overline{w}(x,r)]), \tag{3.2}$$

where

$$\begin{aligned} s[\underline{w}(x,r)] = \underline{W}(v,r) = \int_0^\infty e^{-x}\underline{w}(vx,r)dx, \\ s[\overline{w}(x,r)] = \overline{W}(v,r) = \int_0^\infty e^{-x}\overline{w}(vx,r)dx. \end{aligned} \tag{3.3}$$

The equation (3.1) we can rewrite in the form

$$W(v) = S[w(x)] = \frac{1}{v}(FR)\int_0^\infty e^{-\frac{x}{v}} \odot w(x)dx \tag{3.4}$$

Definition 3.2. *If $k(x)$ and $w(x)$ are fuzzy Riemann integrable functions, then fuzzy convolution of $k(x)$ and $w(x)$ is given by*

$$(k * w)(x) = (FR)\int_0^x k(x-s) \odot w(s)ds \tag{3.5}$$

and the symbol $*$ denotes the fuzzy convolution.

Theorem 3.1. *Let $k : [0,\infty) \to \mathbb{R}$ and $w(x)$ be fuzzy functions. Then the FST of the fuzzy convolution k and w, is given by*

$$S[(k * w)(x)] = vs[k(x)] \odot S[w(x)]. \tag{3.6}$$

We introduce results of FST for fuzzy derivatives.

Theorem 3.2. *Let $w : \mathbb{R} \to E^1$ be a continuous fuzzy-valued function. The functions $e^{-x} \odot w(vx)$, $e^{-x} \odot w^{(n)}(vx)$ are improper fuzzy Riemann-integrable on $[0,\infty)$. Then*

$$S\left[w^{(n)}(x)\right] = \frac{d^n}{dx^n}[S[w(x)]], \tag{3.7}$$

where $S[w(x)]$ denotes the FST of the function w and $n \in \mathbb{N}$.

*Proof.*Let the function $w(x)$ is (i)-differentiable. From definition of FST, we have

$$\begin{aligned} S\left[w^{(n)}(x)\right] &= (FR)\int_0^\infty e^{-x} \odot w^{(n)}(vx)dt = \\ &= \left(\int_0^\infty e^{-x}\underline{w}^{(n)}(vx)dt,\ \int_0^\infty e^{-x}\overline{w}^{(n)}(vx)dt\right) = \\ &= \tfrac{d^n}{dx^n}\left[\int_0^\infty e^{-x}\underline{w}(vx)dx,\ \int_0^\infty e^{-x}\overline{w}(vx)dx\right] = \tfrac{d^n}{dx^n}S[w(x)]. \end{aligned}$$

Theorem 3.3. *Let $w : \mathbb{R} \to E^1$ and the functions $e^{-x} \odot w(vx)$, $e^{-x} \odot w^{(n)}(vx)$ are improper fuzzy Riemann-integrable on $[0,\infty)$ and the functions $w(x),\ w'(x),\ ..., w^{(n-1)}(x)$ be differentiable fuzzy-valued functions and $w(x) = (\underline{w}(x,r), \overline{w}(x,r))$ for all $r \in [0,1]$. Then*

1. *if $w(x)$ and $w'(x),\ ..., w^{(n-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$S\left[w^{(n)}(x)\right] = \left(s\left[\underline{w}^{(n)}(x,r)\right], s\left[\overline{w}^{(n)}(x,r)\right]\right).$$

2. *if $w(x)$ and $w'(x)$, ..., $w^{(n-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$S\left[w^{(n)}(x)\right] = \left(s\left[\overline{w}^{(n)}(x,r)\right], s\left[\underline{w}^{(n)}(x,r)\right]\right) \quad if \;\; n \text{ is even},$$
$$S\left[w^{(n)}(x)\right] = \left(s\left[\underline{w}^{(n)}(x,r)\right], s\left[\overline{w}^{(n)}(x,r)\right]\right) \quad if \;\; n \text{ is odd}.$$

3. *if $w(x)$ is (i)-differentiable and $w'(x)$, ..., $w^{(n-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$S\left[w^{(n)}(x)\right] = \left(s\left[\overline{w}^{(n)}(x,r)\right], s\left[\underline{w}^{(n)}(x,r)\right]\right) \quad if \;\; n \text{ is even},$$
$$S\left[w^{(n)}(x)\right] = \left(s\left[\underline{w}^{(n)}(x,r)\right], s\left[\overline{w}^{(n)}(x,r)\right]\right) \quad if \;\; n \text{ is odd}.$$

4. *if $w(x)$ is (ii)-differentiable and $w'(x)$, ..., $w^{(n-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the n-th order and*

$$S\left[w^{(n)}(x)\right] = \left(s\left[\overline{w}^{(n)}(x,r)\right], s\left[\underline{w}^{(n)}(x,r)\right]\right),$$

where

$$s\left[\underline{w}^{(n)}(x,r)\right] = \frac{1}{v^n} s\left[\underline{w}(x,r)\right] - \sum_{j=1}^{n} \frac{1}{v^j}\left[\underline{w}^{(n-j)}(x,r)\right]_{x=0}, \tag{3.8}$$

$$s\left[\overline{w}^{(n)}(x,r)\right] = \frac{1}{v^n} s\left[\overline{w}(x,r)\right] - \sum_{j=1}^{n} \frac{1}{v^j}\left[\overline{w}^{(n-j)}(x,r)\right]_{x=0} \tag{3.9}$$

*Proof.*Let the function $w(x)$ is (i)-differentiable. By induction we proof the equation (3.8). For $n = 1$ from condition (2.1) we have

$$S\left[w'(x)\right] = (s[\underline{w}'(x,r)], \;\; s\left[\overline{w}'(x,r)\right]).$$

By us part integration and condition (3.3) we obtain

$$s[\underline{w}'(x,r)] = \int_0^{\infty} e^{-x}\underline{w}'(vx,r)dx = \frac{1}{v}(s[\underline{w}(x,r)] - \underline{w}(0,r)).$$

Let for $n = k$ the equation (3.8) holds. Then

$$s\left[\underline{w}^{(k)}(x,r)\right]=\frac{1}{v^k}s\left[\underline{w}(x,r)\right]-\sum_{j=1}^{k}\frac{1}{v^j}\left[\underline{w}^{(k-j)}(x,r)\right]_{x=0}.$$

Hence, for $n=k+1$ we get

$$s\left[\underline{w}^{(k+1)}(x,r)\right]=\tfrac{d}{dx}s\left[\underline{w}^{(k)}(x,r)\right]=\tfrac{1}{v^k}s[\underline{w}'(x,r)]-\sum_{j=1}^{k}\tfrac{1}{v^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0}=$$

$$=\tfrac{1}{v^{k+1}}(s[\underline{w}(x,r)]-s[\underline{w}(0,r)])-\sum_{j=1}^{k}\tfrac{1}{v^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0}=$$

$$=\tfrac{1}{v^{k+1}}s[\underline{w}(x,r)]-\sum_{j=1}^{k+1}\tfrac{1}{v^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0}.$$

4. FUZZY SUMUDU TRANSFORM METHOD FOR SOLVING VOLTERRA FUZZY INTEGRO-DIFFERENTIAL EQUATION

In this section we introduce the parametric form of integro-differential equation (1.1), (1.2) and then apply FST for solving this equation.

Applying FST on both side of integro-differential equation (1.1).

$$\begin{gathered}S\left[(FR)\int_0^x k_1(x-s)\odot w(x)dx\right]\oplus\\ \oplus S\left[(FR)\int_0^x k_2(x-s)\odot w^{(n)}(x)dx\right]=S[g(x)].\end{gathered}\tag{4.1}$$

By using fuzzy convolution (3.6) we obtain

$$vs[k_1(x)]\odot S[w(x)]\oplus vs[k_2(x)]\odot S[w^{(n)}(x)]=S[g(x)].\tag{4.2}$$

Consider the parametric form of $w(x)$ and $g(x)$ respectively $w(x,r)=(\underline{w}(x,r),\overline{w}(x,r))$ and $g(x,r)=(\underline{g}(x,r),\overline{g}(x,r))$, $0\le r\le 1$, $x\in[a,b]$. Let the functions $k_i(x)\ge 0,\ i=1,2$.

Case 1. Let $w^{(n)}(x)=\left(\underline{w}^{(n)}(x,r),\ \overline{w}^{(n)}(x,r)\right)$. Then the parametric form of equation (4.1) is as follows

$$vs[k_1(x)]s[\underline{w}(x,r)]+vs[k_2(x)]s\left[\underline{w}^{(n)}(x,r)\right]=s[\underline{g}(x,r)]$$

$$vs[k_1(x)]s[\overline{w}(x,r)]+vs[k_2(x)]s\left[\overline{w}^{(n)}(x,r)\right]=s[\overline{g}(x,r)].$$

Then, using Theorem 3.3 and the initial conditions we get

$$vs[k_1(x)]s[\underline{w}(x,r)] + vs[k_2(x)]\left[\frac{1}{v^n}s[\underline{w}(x,r)] - \sum_{j=1}^{n}\frac{1}{v^j}\underline{b}_{n-j}(r)\right] = s[\underline{g}(x,r)]$$

$$vs[k_1(x)]s[\overline{w}(x,r)] + vs[k_2(x)]\left[\frac{1}{v^n}s[\overline{w}(x,r)] - \sum_{j=1}^{n}\frac{1}{v^j}\overline{b}_{n-j}(r)\right] = s[\overline{g}(x,r)].$$

Hence, we obtain

$$s[\underline{w}(x,r)] = \frac{v^{n-1}s[\underline{g}(x,r)] + s[k_2(x)]\sum_{j=1}^{n} v^{n-j}\underline{b}_{n-j}(r)}{v^n s[k_1(x)] + s[k_2(x)]} \tag{4.3}$$

$$s[\underline{w}(x,r)] = \frac{v^{n-1}s[\underline{g}(x,r)] + s[k_2(x)]\sum_{j=1}^{n} v^{n-j}\underline{b}_{n-j}(r)}{v^n s[k_1(x)] + s[k_2(x)]} \tag{4.4}$$

Applying the inverse Sumudu transform we find the solution of the equation (1.1), (1.2).

Case 2. Let $w^{(n)}(x) = \left(\overline{w}^{(n)}(x,r),\ \underline{w}^{(n)}(x,r)\right)$. Then the parametric form of equation (4.1) is as follows

$$vs[k_1(x)]s[\underline{w}(x,r)] + vs[k_2(x)]s\left[\overline{w}^{(n)}(x,r)\right] = s[\underline{g}(x,r)]$$

$$vs[k_1(x)]s[\overline{w}(x,r)] + vs[k_2(x)]s\left[\underline{w}^{(n)}(x,r)\right] = s[\overline{g}(x,r)].$$

Then, using Theorem 3.3 and the initial conditions we get

$$vs[k_1(x)]s[\underline{w}(x,r)] + vs[k_2(x)]\left[\frac{1}{v^n}s[\overline{w}(x,r)] - \sum_{j=1}^{n}\frac{1}{v^j}\overline{b}_{n-j}(r)\right] = s[\underline{g}(x,r)]$$

$$vs[k_1(x)]s[\overline{w}(x,r)] + vs[k_2(x)]\left[\frac{1}{v^n}s[\underline{w}(x,r)] - \sum_{j=1}^{n}\frac{1}{v^j}\underline{b}_{n-j}(r)\right] = s[\overline{g}(x,r)].$$

Hence, we obtain

$$\begin{aligned} &v^n s[k_1(x)]s[\underline{w}(x,r)] + s[k_2(x)]s[\overline{w}(x,r)] = \\ &= v^{n-1}s[\underline{g}(x,r)] + s[k_2(x)]\sum_{j=1}^{n} v^{n-j}\overline{b}_{n-j}(r) \end{aligned} \tag{4.5}$$

$$\begin{aligned}&v^n s[k_1(x)]s[\overline{w}(x,r)] + s[k_2(x)]s[\underline{w}(x,r)] = \\ &= v^{n-1}s[\overline{g}(x,r)] + s[k_2(x)]\sum_{j=1}^{n} v^{n-j}\underline{b}_{n-j}(r).\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we find $s[\underline{w}(x,r)]$ and $s[\overline{w}(x,r)]$. Applying the inverse Sumudu transform we obtain the solution of the equation (1.1), (1.2).

5. NUMERICAL EXPERIMENT

In this section, we find the solution of VFIDE by using FST. Consider the following equation

$$\int_0^x w(s)ds \oplus (FR)\int_0^x (x-s)\odot w''(s)ds = g(x), \quad x\in[0,1], \quad r\in[0,1],$$

with initial condition $w(0)=(0,0)$, $w'(0)=(0,0)$ and

$$g(x) = \left(\left(\frac{x^3}{3}+x^2\right)(1+r), \left(\frac{x^3}{3}+x^2\right)(3-r)\right).$$

In this case $k_1(x-s)=1$, $k_2(x-s)=x-s$ and $b_0=(0,0)$, $b_1=(0,0)$.

Case 1. Let $w(x)$ and $w'(x)$ are (i) or (ii)-differentiable. Then $w''(x)=(\underline{w}''(x,r), \overline{w}''(x,r))$. Using the equations (4.3) and (4.4) we obtain

$$s\left[\underline{w}(x,r)\right] = \frac{vs[\underline{g}(x,r)]}{v^2 s[k_1(x)]+s[k_2(x)]}, \quad s\left[\overline{w}(x,r)\right] = \frac{vs[\overline{g}(x,r)]}{v^2 s[k_1(x)]+s[k_2(x)]},$$

where

$$s[\underline{g}(x,r)] = \left(\frac{1}{3}s[x^3]+s[x^2]\right)(1+r) = \left(2v^3+2v^2\right)(1+r),$$

$$s[\overline{g}(x,r)] = \left(\frac{1}{3}s[x^3]+s[x^2]\right)(3-r) = \left(2v^3+2v^2\right)(3-r),$$

$$s[k_1(x)] = s[1] = 1, \quad s[k_2(x)] = s[x] = v$$

Hence

$$s\left[\underline{w}(x,r)\right] = \frac{v\left(2v^3+2v^2\right)(1+r)}{v^2+v} = \frac{2v^3(1+v)(1+r)}{v(1+v)} = 2v^2(1+r),$$

$$s\left[\overline{w}(x,r)\right] = \frac{v\left(2v^3+2v^2\right)(3-r)}{v^2+v} = \frac{2v^3(1+v)(3-r)}{v(1+v)} = 2v^2(3-r),$$

Applying the inverse Sumudu transform we obtain

$$\underline{w}(x,r) = s^{-1}[2v^2(1+r)] = x^2(1+r), \ \ \overline{w}(x,r) = s^{-1}[2v^2(3-r)] = x^2(3-r).$$

Case 2. Let $w(x)$ is (i)-differentiable and $w'(x)$ is (ii)-differentiable or $w(x)$ is (ii)-differentiable and $w'(x)$ is (i)-differentiable. Then $w''(x) = (\overline{w}''(x,r), \ \underline{w}''(x,r))$. Using the equations (4.5) and (4.6) we obtain

$$vs[\underline{w}(x,r)] + s[\overline{w}(x,r)] = \left(2v^3 + 2v^2\right)(1+r),$$

$$vs[\overline{w}(x,r)] + s[\underline{w}(x,r)] = \left(2v^3 + 2v^2\right)(3-r).$$

Hence

$$s\left[\underline{w}(x,r)\right] = -8\left(v+1-\frac{1}{1-v}\right) + 2\left(v^2+2v+2-\frac{2}{1-v}\right)(1+r),$$

$$s\left[\overline{w}(x,r)\right] = -8\left(v+1-\frac{1}{1-v}\right) + 2\left(v^2+2v+2-\frac{2}{1-v}\right)(3-r).$$

Applying the inverse Sumudu transform we obtain

$$\begin{aligned} \underline{w}(x,r) &= -8s^{-1}\left[v+1-\tfrac{1}{1-v}\right] + s^{-1}\left[2v^2+4v+4-\tfrac{4}{1-v}\right](1+r) = \\ &= -8(x+1-e^x) + (x^2+4x+4-4e^x)(1+r), \\ \overline{w}(x,r) &= -8s^{-1}\left[v+1-\tfrac{1}{1-v}\right] + s^{-1}\left[2v^2+4v+4-\tfrac{4}{1-v}\right](3-r) = \\ &= -8(x+1-e^x) + (x^2+4x+4-4e^x)(3-r). \end{aligned}$$

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