International Journal of Differential Equations and Applications
Volume **23**, No. 1 (2024), pages: 105-121
ISSN (Print): 1311-2872; ISSN (Online): 1314-6084;
url: https://www.ijdea.eu

ijdea.eu

ON THE DYNAMICS OF AN EXTENDED ANHARMONIC OSCILLATOR: APPLICATIONS

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ABSTRACT: Using Melnikov's method, the existence of chaotic behaviour in the sense of Smale in an extended anharmonic oscillator is established. We demonstrate some modules for investigating the dynamics of the proposed model. This will be included as an integral part of a planned much more general Web-based application for scientific computing. One possible application that Melnikov functions may find in the modeling of radiating antenna patterns is considered.

Key Words: extended anharmonic oscillator, hetero-clinic orbit, Melnikov function, radiating antenna array

Received: October 11, 2024 **Revised:** December 10, 2024
Published: December 19, 2024 **doi:** 10.12732/ijdea.v23i1.9

1. INTRODUCTION

Chaotic behaviour of some dynamical systems can be explained by the existence of transverse homoclinic points. Well-known examples of the application of Melnikov's method [1] to investigations of chaotic behaviour in planar autonomous systems analyse the dynamics of the simple pendulum (Holmes [2], Holmes and Marsden [3], Marsden [4]) and the Duffing oscillator (Guckenheimer and Holmes [5], Holmes [6], Lichtenberg and Lieberman [7], Wiggins [8]). The publications on this topic are significant and varied (see for example [9]–[12]). In [13], the authors presented the following model

$$\begin{cases} \dfrac{dx}{dt} &= a^2x - 2y(x^2+y^2) + \epsilon\,(cx + b\sin(\omega t)) \\ \dfrac{dy}{dt} &= -a^2y + 2x(x^2+y^2) \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$ is a perturbation parameter, $a > 0$, $\omega > 0$ and b and c are real numbers and demonstrate, by an application of the Melnikov technique, the existence of transverse homo-clinic points for the Poincare' map of the time–periodically perturbed system. Some computer simulations of the chaotic dynamics of the system for different choices of parameters are discussed. In [14], the same system is considered and a condition for the existence of subharmonic orbits in the perturbed system is deduced, using the subharmonic Melnikov theory. In this paper, we suggest a new class of extended anharmonic oscillator of the type (1). Investigations in the light of Melnikov's approach is considered. Several simulations are composed. We demonstrate some modules for investigating the dynamics of the proposed model. This will be included as an integral part of a planned much more general Web-based application for scientific computing. One possible application that Melnikov functions may find in the modeling and synthesis of radiating antenna patterns is also considered.

2. THE NEW MODEL. MELNIKOV'S APPROACH

In this paper, we suggest a modified model of the type:

$$\begin{cases} \dfrac{dx}{dt} &= a^2x - 2y(x^2+y^2) + \epsilon\left(cx + b\sin(\omega t) + \displaystyle\sum_{j=2}^{N} b_{j-1}\sin(j\omega t)\right) \\ \dfrac{dy}{dt} &= -a^2y + 2x(x^2+y^2) \end{cases} \tag{2}$$

where $0 \leq \epsilon < 1$, $a > 0$, $\omega > 0$ and b and c are real numbers; $b_i \geq 0$; $i = 1, 2, \ldots, N$ and N is integer. In the special case $N = 1$, our model (2) coincides with the classical model (1). Let $(x_0(t), y_0(t))$ is one of the loops of the homo-clinic lemniscate given by (see for more details precise considerations in [13])

$$x_0(t) = \frac{\sqrt{2}ae^{a^2t}}{1+e^{4a^2t}};\ \ y_0(t) = \frac{\sqrt{2}ae^{3a^2t}}{1+e^{4a^2t}};\ \ t \in R.$$

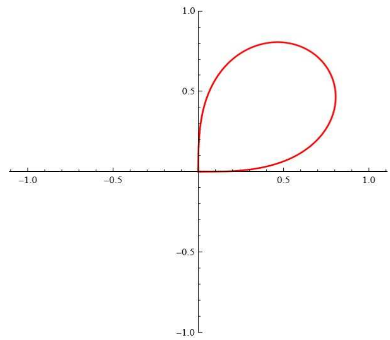

Figure 1: the orbit $q_0(t) = (x_0(t), y_0(t))$.

See also Fig. 1 for fixed $a = 1$. The Melnikov function [1] corresponding to the system (2) is of the form:

$$M(t_0) = \int_{-\infty}^{\infty} (-\frac{dy_0(s)}{ds}(b\sin(\omega(s+t_0)) + \sum_{j=2}^{N} b_{j-1}\sin(j\omega(s+t_0))) \\ +a^2cx_0(s)y_0(s) - 2cx_0^4(s) - 2cx_0^2(s)y_0^2(s))ds \quad (3)$$

The Melnikov function can be represented in another way, using the following familiar equality

$$sin(i\omega(t+t_0)) = sin(i\omega t)cos(i\omega t_0) + cos(i\omega t)sin(i\omega t_0).$$

We will not dwell on this question here. For us, it is more important to note that such a representation is more appropriate and can be used directly by users of the corresponding specialized module implemented for example in CAS Mathematica. If $M(t_0) = 0$ and $\frac{M(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then chaos occurs.

2.1. THE CASE $N = 2$, $A = 1$

From (3) we find:

Proposition 1. If $N = 2$, $a = 1$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation $M(t_0) = 0$ (see Figure 2).

For example for $N = 2$, $a = 1$, $\omega = 0.5$, $b = 0.36$, $c = 0.085$, $b_1 = 0.29$ the Melnikov function $M(t_0)$ is depicted in Figure 3.

2.2. THE CASE $N = 3$, $A = 1$

From (3) we find:

$$\frac{1}{16} e^{-2 i t0 \omega} \Big(-8 c e^{2 i t0 \omega} + \sqrt{2}\, b e^{3 i t0 \omega} \pi \omega \mathrm{Cot}\Big[\frac{1}{8} \pi (3 + i \omega)\Big] + 2\sqrt{2}\, b1 e^{4 i t0 \omega} \pi \omega \mathrm{Cot}\Big[\frac{1}{8} \pi (3 + 2 i \omega)\Big] + \sqrt{2}\, b e^{i t0 \omega} \pi \omega \mathrm{Cot}\Big[\frac{1}{8} (\pi + i \pi \omega)\Big] +$$
$$2\sqrt{2}\, b1 \pi \omega \mathrm{Cot}\Big[\frac{1}{8} (\pi + 2 i \pi \omega)\Big] + \sqrt{2}\, b e^{3 i t0 \omega} \pi \omega \mathrm{Tan}\Big[\frac{1}{8} \pi (3 + i \omega)\Big] + 2\sqrt{2}\, b1 e^{4 i t0 \omega} \pi \omega \mathrm{Tan}\Big[\frac{1}{8} \pi (3 + 2 i \omega)\Big] +$$
$$\sqrt{2}\, b e^{i t0 \omega} \pi \omega \mathrm{Tan}\Big[\frac{1}{8} (\pi + i \pi \omega)\Big] + 2\sqrt{2}\, b1 \pi \omega \mathrm{Tan}\Big[\frac{1}{8} (\pi + 2 i \pi \omega)\Big]\Big) = 0$$

Figure 2: $M(t_0) = 0$ (generating using CAS Mathematica (Proposition 1)).

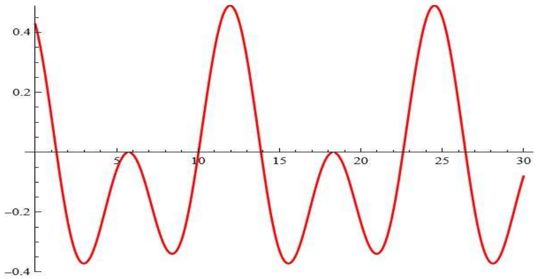

Figure 3: The equation $M(t_0) = 0$ for $N = 2$, $a = 1$, $\omega = 0.5$, $b = 0.36$, $c = 0.085$, $b_1 = 0.29$; (from Proposition 1).

Proposition 2. If $N = 3$, $a = 1$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation $M(t_0) = 0$ (see Figure 4)

For example for $N = 3$, $a = 1$, $\omega = 0.25$, $b = 0.41$, $c = 0.001$, $b_1 = 0.39$, $b_2 = 0.29$ the Melnikov function $M(t_0) = 0$ is depicted in Fig. 5.

From the above examples, the reader can himself formulate the corresponding Melnikov criterion for the occurrence of chaos in the considered dynamic system for some N.

3. SOME SIMULATIONS. APPLICATIONS

We will look at some interesting simulations on model (2):

Example 1. For given $N = 3$, $\omega = 0.5$, $a = 0.5$, $c = 0.085$, $\epsilon = 0.03$, $b = 0.46$, $b_1 = 0.02$, $b_2 = 0.4$ the simulations on the system (2) for $x_0 = 0.7$; $y_0 = 0.4$ are depicted on Fig. 6.

Example 2. For given $N = 3$, $\omega = 0.5$, $a = 1$, $c = 0.085$, $\epsilon = 0.03$, $b = 0.46$, $b_1 = 0.02$, $b_2 = 0.4$ the simulations on the system (2) for $x_0 = 0.3$; $y_0 = 0.1$ are depicted on Fig. 7.

$$
\begin{aligned}
&-\frac{c}{2}-\frac{i\, b\, e^{-i t0 \omega}}{2\sqrt{2}}+\frac{i\, b\, e^{i t0 \omega}}{2\sqrt{2}}-\frac{i\, b1\, e^{-2 i t0 \omega}}{2\sqrt{2}}+\frac{i\, b1\, e^{2 i t0 \omega}}{2\sqrt{2}}-\frac{i\, b2\, e^{-3 i t0 \omega}}{64\sqrt{2}}+\frac{i\, b2\, e^{3 i t0 \omega}}{64\sqrt{2}}+\frac{b1\, e^{-2 i t0 \omega}(21 i-2\omega)}{32\sqrt{2}}-\frac{i\, b2\, e^{-3 i t0 \omega}(5-3 i \omega)}{32\sqrt{2}}+\frac{3 i\, b2\, e^{3 i t0 \omega}(7-3 i \omega)}{64\sqrt{2}}+\frac{i\, b2\, e^{3 i t0 \omega}(5+3 i \omega)}{32\sqrt{2}}-\\
&\frac{3 i\, b2\, e^{-3 i t0 \omega}(7+3 i \omega)}{64\sqrt{2}}-\frac{3\, b2\, e^{-3 i t0 \omega}\omega}{64\sqrt{2}}-\frac{3\, b2\, e^{3 i t0 \omega}\omega}{64\sqrt{2}}+\frac{3\, b\, e^{-i t0 \omega}(-i+\omega)}{32\sqrt{2}}+\frac{3\, b2\, e^{3 i t0 \omega}(-i+\omega)}{32\sqrt{2}}+\frac{3\, b\, e^{i t0 \omega}(i+\omega)}{32\sqrt{2}}+\frac{3\, b2\, e^{-3 i t0 \omega}(i+\omega)}{32\sqrt{2}}+\frac{b\, e^{i t0 \omega}(-3 i+\omega)}{32\sqrt{2}}+\frac{b\, e^{-i t0 \omega}(3 i+\omega)}{32\sqrt{2}}-\\
&\frac{3\, b2\, e^{-3 i t0 \omega}(-7 i+\omega)}{32\sqrt{2}}-\frac{3\, b2\, e^{3 i t0 \omega}(7 i+\omega)}{32\sqrt{2}}-\frac{b\, e^{-i t0 \omega}(-21 i+\omega)}{32\sqrt{2}}-\frac{b\, e^{i t0 \omega}(21 i+\omega)}{32\sqrt{2}}+\frac{3\, b1\, e^{-2 i t0 \omega}(-i+2\omega)}{32\sqrt{2}}+\frac{3\, b1\, e^{2 i t0 \omega}(i+2\omega)}{32\sqrt{2}}+\frac{b1\, e^{2 i t0 \omega}(-3 i+2\omega)}{32\sqrt{2}}+\frac{b1\, e^{-2 i t0 \omega}(3 i+2\omega)}{32\sqrt{2}}-\\
&\frac{b1\, e^{2 i t0 \omega}(21 i+2\omega)}{32\sqrt{2}}+\frac{3\, b2\, e^{-3 i t0 \omega}(-i+3\omega)}{32\sqrt{2}}+\frac{3\, b2\, e^{3 i t0 \omega}(i+3\omega)}{32\sqrt{2}}-\frac{b\, e^{i t0 \omega}(-5 i+3\omega)}{32\sqrt{2}}-\frac{b\, e^{-i t0 \omega}(5 i+3\omega)}{32\sqrt{2}}-\frac{b1\, e^{2 i t0 \omega}(-5 i+6\omega)}{32\sqrt{2}}-\frac{b1\, e^{-2 i t0 \omega}(5 i+6\omega)}{32\sqrt{2}}-\frac{b2\, e^{3 i t0 \omega}(-5 i+9\omega)}{32\sqrt{2}}-\\
&\frac{b2\, e^{-3 i t0 \omega}(5 i+9\omega)}{32\sqrt{2}}+\frac{b\, e^{i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}-\frac{i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{b\, e^{-i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}+\frac{i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{b1\, e^{2 i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}-\frac{i\pi\omega}{4}\right]}{4\sqrt{2}}+\frac{b1\, e^{-2 i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}+\frac{i\pi\omega}{4}\right]}{4\sqrt{2}}+\frac{3\, b2\, e^{3 i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}-\frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{3\, b2\, e^{-3 i t0 \omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8}+\frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}}+\\
&\frac{b\, e^{i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}-\frac{i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{b\, e^{-i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}+\frac{i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{b1\, e^{2 i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}-\frac{i\pi\omega}{4}\right]}{4\sqrt{2}}+\frac{b1\, e^{-2 i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}+\frac{i\pi\omega}{4}\right]}{4\sqrt{2}}+\frac{3\, b2\, e^{3 i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}-\frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}}+\frac{3\, b2\, e^{-3 i t0 \omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8}+\frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}}=0
\end{aligned}
$$

Figure 4: $M(t_0) = 0$ (generating using CAS Mathematica (Proposition 2)).

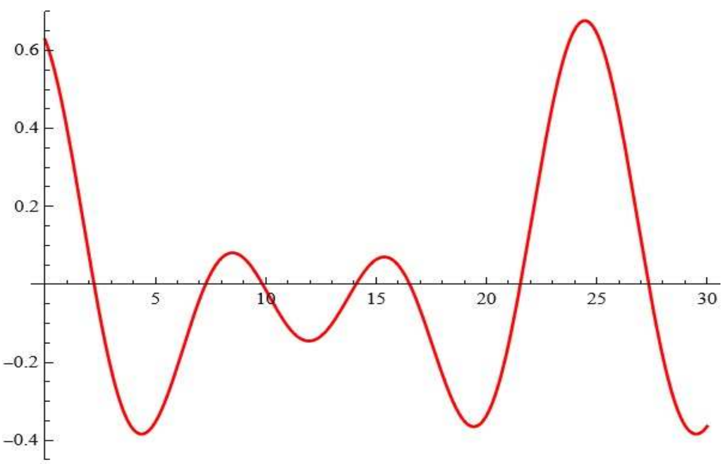

Figure 5: The equation $M(t_0) = 0$ for $N = 3$, $a = 1$, $\omega = 0.25$, $b = 0.41$, $c = 0.001$, $b_1 = 0.39$, $b_2 = 0.29$ (from Proposition 2).

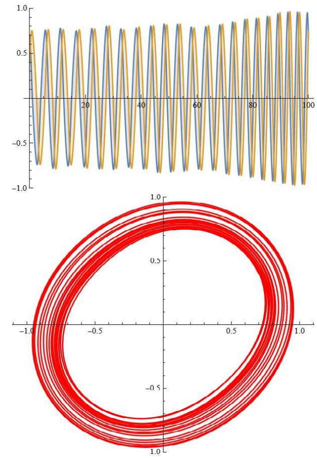

Figure 6: a) the solutions of the system (2); b) phase space (Example 1).

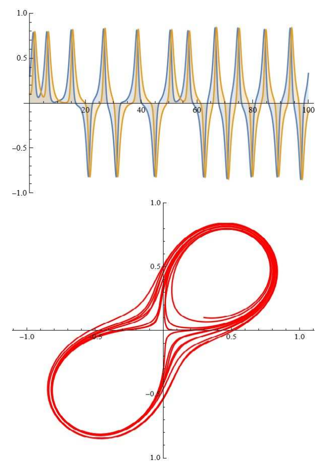

Figure 7: a) the solutions of the system (2); b) phase space (Example 2).

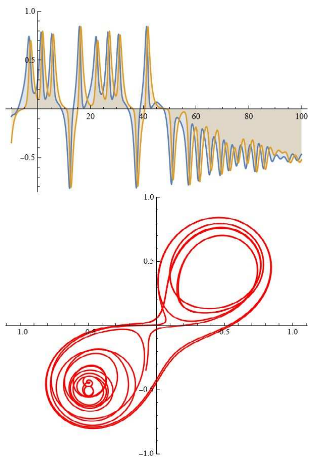

Figure 8: a) the solutions of the system (2); b) phase space (Example 3).

Example 3. For given $N = 4$, $\omega = 0.5$, $a = 1$, $c = -1$, $\epsilon = 0.08$, $b = 1$, $b_1 = 0.02$, $b_2 = 0.4$, $b_3 = 0.1$ the simulations on the system (2) for $x_0 = 0.3$; $y_0 = 0.1$ are depicted on Fig. 8.

Let us now focus on $M(t)$.

In our previous publications, we discussed a possible application of the Melnikov functions can find in modeling and synthesis of radiation antenna diagrams. For more details see for example [22]– [25].

We define the hypothetical normalized antenna factor as follows: $M^*(\theta) = \frac{1}{D}|M(K\cos\theta + k_1)|$ where θ is the azimuth angle; $K = kd$; $k = \frac{2\pi}{\lambda}$; λ is the wave length; d is the distance between emitters; k_1 is the phase difference.

Let's focus on the case when the user fixes, for example, $N = 4$.

From (3) we find:

Proposition 3. If $N = 4$, $a = 1$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation $M(t_0) = 0$ (see Figure 9)

Example 4. For fixed $N = 4$, $k = 8$, $k_1 = -0.6$, $\omega = 0.24$, $a = 1$, $b = 0.41$, $c = 0.001$, $b_1 = 0.39$, $b_2 = 0.29$, $b_3 = 0.19$, Melnikov function and Melnikov antenna factor are depicted in Figure 10.

Example 5. For fixed $N = 6$, $k = 9$, $k_1 = -0.69$, $\omega = 0.16$, $a = 1$, $b = 0.41$, $c = 0.001$, $b_1 = 0.39$, $b_2 = 0.29$, $b_3 = 0.19$, $b_4 = 0.12$, $b_5 = 0.22$, Melnikov function and Melnikov antenna factor are depicted in Figure 11.

Example 6. For fixed $N = 8$, $k = 9.5$, $k_1 = -0.6$, $\omega = 0.16$, $a = 1$, $b = 0.31$, $c = 0.001$, $b_1 = 0.39$, $b_2 = 0.29$, $b_3 = 0.19$, $b_4 = 0.12$, $b_5 = 0.22$ $b_6 = 0.15$, $b_7 = 0.28$,

$$
\begin{aligned}
&-\frac{c}{2} - \frac{361\, i\, b\, e^{-i t\omega}}{768\sqrt{2}} - \frac{i\, b\, e^{i t\omega}}{24\sqrt{2}} - \frac{11\, i\, b1\, e^{-2 i t\omega}}{64\sqrt{2}} + \frac{15\, i\, b1\, e^{2 i t\omega}}{128\sqrt{2}} - \frac{187\, i\, b2\, e^{-3 i t\omega}}{768\sqrt{2}} + \frac{5\, i\, b2\, e^{3 i t\omega}}{48\sqrt{2}} - \frac{293\, i\, b3\, e^{-4 i t\omega}}{768\sqrt{2}} + \frac{41\, i\, b3\, e^{4 i t\omega}}{256\sqrt{2}} - \frac{i\, b3\, e^{-4 i t\omega}(5 i - 4\omega)^2}{64\sqrt{2}} - \frac{i\, b3\, e^{4 i t\omega}(7 i - 4\omega)^2}{192\sqrt{2}} - \frac{i\, b2\, e^{-3 i t\omega}(5 i - 3\omega)^2}{64\sqrt{2}} - \\
&\frac{i\, b2\, e^{3 i t\omega}(7 i - 3\omega)^2}{192\sqrt{2}} + \frac{25\, b\, e^{-i t\omega}\omega}{192\sqrt{2}} - \frac{b\, e^{i t\omega}\omega}{192\sqrt{2}} + \frac{5\, b1\, e^{-2 i t\omega}\omega}{32\sqrt{2}} + \frac{3\, b1\, e^{2 i t\omega}\omega}{32\sqrt{2}} + \frac{5\, b2\, e^{-3 i t\omega}\omega}{32\sqrt{2}} + \frac{15\, b2\, e^{3 i t\omega}\omega}{64\sqrt{2}} + \frac{9\, b3\, e^{-4 i t\omega}\omega}{16\sqrt{2}} + \frac{b3\, e^{4 i t\omega}\omega}{4\sqrt{2}} + \frac{11\, i\, b\, e^{-i t\omega}\omega^2}{768\sqrt{2}} - \frac{i\, b\, e^{i t\omega}\omega^2}{192\sqrt{2}} + \frac{i\, b1\, e^{-2 i t\omega}\omega^2}{24\sqrt{2}} - \frac{5\, i\, b1\, e^{2 i t\omega}\omega^2}{96\sqrt{2}} + \\
&\frac{39\, i\, b2\, e^{-3 i t\omega}\omega^2}{256\sqrt{2}} - \frac{9\, i\, b2\, e^{3 i t\omega}\omega^2}{64\sqrt{2}} + \frac{5\, i\, b3\, e^{-4 i t\omega}\omega^2}{16\sqrt{2}} - \frac{13\, i\, b3\, e^{4 i t\omega}\omega^2}{48\sqrt{2}} - \frac{i\, b\, e^{-i t\omega}(-5 i + \omega)^2}{64\sqrt{2}} + \frac{i\, b\, e^{i t\omega}(5 i + \omega)^2}{64\sqrt{2}} - \frac{i\, b\, e^{i t\omega}(-7 i + \omega)^2}{192\sqrt{2}} + \frac{i\, b\, e^{-i t\omega}(7 i + \omega)^2}{192\sqrt{2}} + \frac{i\, b1\, e^{2 i t\omega}(5 i + 2\omega)^2}{64\sqrt{2}} + \frac{i\, b1\, e^{-2 i t\omega}(7 i + 2\omega)^2}{192\sqrt{2}} + \\
&\frac{i\, b2\, e^{3 i t\omega}(5 i + 3\omega)^2}{64\sqrt{2}} + \frac{i\, b2\, e^{-3 i t\omega}(7 i + 3\omega)^2}{192\sqrt{2}} + \frac{i\, b3\, e^{4 i t\omega}(5 i + 4\omega)^2}{64\sqrt{2}} + \frac{i\, b3\, e^{-4 i t\omega}(7 i + 4\omega)^2}{192\sqrt{2}} + \frac{i\, b2\, e^{3 i t\omega}\left(43 + 36 i\omega - 9\omega^2\right)}{768\sqrt{2}} - \frac{i\, b2\, e^{-3 i t\omega}\left(71 + 48 i\omega - 9\omega^2\right)}{256\sqrt{2}} - \frac{i\, b1\, e^{-2 i t\omega}\left(71 + 32 i\omega - 4\omega^2\right)}{256\sqrt{2}} + \\
&\frac{b\, e^{-i t\omega}\left(15 i + 8\omega - i\omega^2\right)}{192\sqrt{2}} + \frac{b\, e^{-i t\omega}\left(24 i + 5\omega + i\omega^2\right)}{96\sqrt{2}} + \frac{b\, e^{i t\omega}\left(-15 i + 8\omega + i\omega^2\right)}{192\sqrt{2}} + \frac{b\, e^{i t\omega}\left(21 i + 20\omega + i\omega^2\right)}{192\sqrt{2}} + \frac{b1\, e^{2 i t\omega}\left(-12 i + 5\omega - 2 i\omega^2\right)}{48\sqrt{2}} + \frac{b2\, e^{3 i t\omega}\left(-8 i + 5\omega - 3 i\omega^2\right)}{32\sqrt{2}} + \frac{b2\, e^{-3 i t\omega}\left(5 i + 8\omega - 3 i\omega^2\right)}{64\sqrt{2}} + \\
&\frac{b2\, e^{-3 i t\omega}\left(-7 i + 20\omega - 3 i\omega^2\right)}{64\sqrt{2}} + \frac{b2\, e^{-3 i t\omega}\left(8 i + 5\omega + 3 i\omega^2\right)}{32\sqrt{2}} + \frac{b2\, e^{3 i t\omega}\left(-5 i + 8\omega + 3 i\omega^2\right)}{64\sqrt{2}} + \frac{b2\, e^{3 i t\omega}\left(7 i + 20\omega + 3 i\omega^2\right)}{64\sqrt{2}} + \frac{b3\, e^{4 i t\omega}\left(-6 i + 5\omega - 4 i\omega^2\right)}{24\sqrt{2}} + \frac{b1\, e^{-2 i t\omega}\left(15 i + 16\omega - 4 i\omega^2\right)}{192\sqrt{2}} + \frac{b1\, e^{-2 i t\omega}\left(-21 i + 40\omega - 4 i\omega^2\right)}{192\sqrt{2}} + \\
&\frac{b3\, e^{-4 i t\omega}\left(6 i + 5\omega + 4 i\omega^2\right)}{24\sqrt{2}} + \frac{b1\, e^{2 i t\omega}\left(-15 i + 16\omega + 4 i\omega^2\right)}{192\sqrt{2}} + \frac{b1\, e^{2 i t\omega}\left(21 i + 40\omega + 4 i\omega^2\right)}{192\sqrt{2}} + \frac{b3\, e^{-4 i t\omega}\left(15 i + 32\omega - 16 i\omega^2\right)}{192\sqrt{2}} + \frac{b3\, e^{-4 i t\omega}\left(-21 i + 80\omega - 16 i\omega^2\right)}{192\sqrt{2}} + \frac{b3\, e^{4 i t\omega}\left(-15 i + 32\omega + 16 i\omega^2\right)}{192\sqrt{2}} + \\
&\frac{b3\, e^{4 i t\omega}\left(21 i + 80\omega + 16 i\omega^2\right)}{192\sqrt{2}} + \frac{i\, b\, e^{-i t\omega}\left(-7 + \omega^2\right)}{768\sqrt{2}} - \frac{i\, b\, e^{i t\omega}\left(-7 + \omega^2\right)}{768\sqrt{2}} + \frac{5\, i\, b\, e^{-i t\omega}\left(-11 + 4 i\omega + \omega^2\right)}{768\sqrt{2}} - \frac{i\, b\, e^{i t\omega}\left(-11 + 4 i\omega + \omega^2\right)}{256\sqrt{2}} - \frac{i\, b\, e^{i t\omega}\left(24 + 5 i\omega + \omega^2\right)}{96\sqrt{2}} - \frac{i\, b\, e^{-i t\omega}\left(-23 + 8 i\omega + \omega^2\right)}{768\sqrt{2}} - \frac{5\, i\, b\, e^{i t\omega}\left(-23 + 8 i\omega + \omega^2\right)}{768\sqrt{2}} - \\
&\frac{i\, b\, e^{-i t\omega}\left(-8 - 9 i\omega + \omega^2\right)}{96\sqrt{2}} + \frac{i\, b\, e^{i t\omega}\left(-8 + 9 i\omega + \omega^2\right)}{96\sqrt{2}} + \frac{i\, b\, e^{-i t\omega}\left(-43 + 12 i\omega + \omega^2\right)}{768\sqrt{2}} - \frac{i\, b\, e^{i t\omega}\left(-71 + 16 i\omega + \omega^2\right)}{256\sqrt{2}} - \frac{i\, b\, e^{-i t\omega}\left(-107 + 20 i\omega + \omega^2\right)}{768\sqrt{2}} - \frac{i\, b\, e^{-i t\omega}\left(21 + 20 i\omega + \omega^2\right)}{192\sqrt{2}} - \frac{i\, b1\, e^{-2 i t\omega}\left(-4 - 9 i\omega + 2\omega^2\right)}{48\sqrt{2}} + \\
&\frac{5\, i\, b1\, e^{-2 i t\omega}\left(-11 + 8 i\omega + 4\omega^2\right)}{768\sqrt{2}} - \frac{i\, b1\, e^{2 i t\omega}\left(-11 + 8 i\omega + 4\omega^2\right)}{256\sqrt{2}} - \frac{i\, b3\, e^{-4 i t\omega}\left(-2 - 9 i\omega + 4\omega^2\right)}{24\sqrt{2}} + \frac{i\, b3\, e^{4 i t\omega}\left(-2 + 9 i\omega + 4\omega^2\right)}{24\sqrt{2}} - \frac{i\, b1\, e^{-2 i t\omega}\left(-5 - 12 i\omega + 4\omega^2\right)}{128\sqrt{2}} - \frac{i\, b1\, e^{2 i t\omega}\left(-21 - 20 i\omega + 4\omega^2\right)}{384\sqrt{2}} + \frac{i\, b1\, e^{-2 i t\omega}\left(-43 + 24 i\omega + 4\omega^2\right)}{768\sqrt{2}} - \\
&\frac{i\, b1\, e^{2 i t\omega}\left(-71 + 32 i\omega + 4\omega^2\right)}{256\sqrt{2}} + \frac{i\, b1\, e^{2 i t\omega}\left(15 + 36 i\omega + 4\omega^2\right)}{128\sqrt{2}} - \frac{i\, b1\, e^{-2 i t\omega}\left(-107 + 40 i\omega + 4\omega^2\right)}{768\sqrt{2}} + \frac{i\, b1\, e^{-2 i t\omega}\left(231 + 44 i\omega + 4\omega^2\right)}{384\sqrt{2}} - \frac{i\, b2\, e^{-3 i t\omega}\left(-8 - 27 i\omega + 9\omega^2\right)}{96\sqrt{2}} + \frac{i\, b2\, e^{3 i t\omega}\left(-8 + 27 i\omega + 9\omega^2\right)}{96\sqrt{2}} + \frac{i\, b2\, e^{-3 i t\omega}\left(-43 + 36 i\omega + 9\omega^2\right)}{768\sqrt{2}} - \\
&\frac{i\, b2\, e^{3 i t\omega}\left(-71 + 48 i\omega + 9\omega^2\right)}{256\sqrt{2}} - \frac{i\, b2\, e^{-3 i t\omega}\left(-107 + 60 i\omega + 9\omega^2\right)}{768\sqrt{2}} + \frac{i\, b3\, e^{-4 i t\omega}\left(-43 + 48 i\omega + 16\omega^2\right)}{768\sqrt{2}} - \frac{i\, b3\, e^{4 i t\omega}\left(-71 + 64 i\omega + 16\omega^2\right)}{256\sqrt{2}} + \frac{b\, e^{i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} - \frac{i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b\, e^{-i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} + \frac{i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b1\, e^{2 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} - \frac{i\pi\omega}{4}\right]}{4\sqrt{2}} + \\
&\frac{b1\, e^{-2 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} + \frac{i\pi\omega}{4}\right]}{4\sqrt{2}} + \frac{3\, b2\, e^{3 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} - \frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{3\, b2\, e^{-3 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} + \frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b3\, e^{4 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} - \frac{i\pi\omega}{2}\right]}{2\sqrt{2}} + \frac{b3\, e^{-4 i t\omega}\pi\omega\,\mathrm{Cot}\left[\frac{\pi}{8} + \frac{i\pi\omega}{2}\right]}{2\sqrt{2}} + \frac{b\, e^{i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} - \frac{i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b\, e^{-i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} + \frac{i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b1\, e^{2 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} - \frac{i\pi\omega}{4}\right]}{4\sqrt{2}} + \\
&\frac{b1\, e^{-2 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} + \frac{i\pi\omega}{4}\right]}{4\sqrt{2}} + \frac{3\, b2\, e^{3 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} - \frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{3\, b2\, e^{-3 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} + \frac{3 i\pi\omega}{8}\right]}{8\sqrt{2}} + \frac{b3\, e^{4 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} - \frac{i\pi\omega}{2}\right]}{2\sqrt{2}} + \frac{b3\, e^{-4 i t\omega}\pi\omega\,\mathrm{Tan}\left[\frac{\pi}{8} + \frac{i\pi\omega}{2}\right]}{2\sqrt{2}} = 0
\end{aligned}
$$

Figure 9: $M(t_0) = 0$ (generating using CAS Mathematica (Proposition 3)).

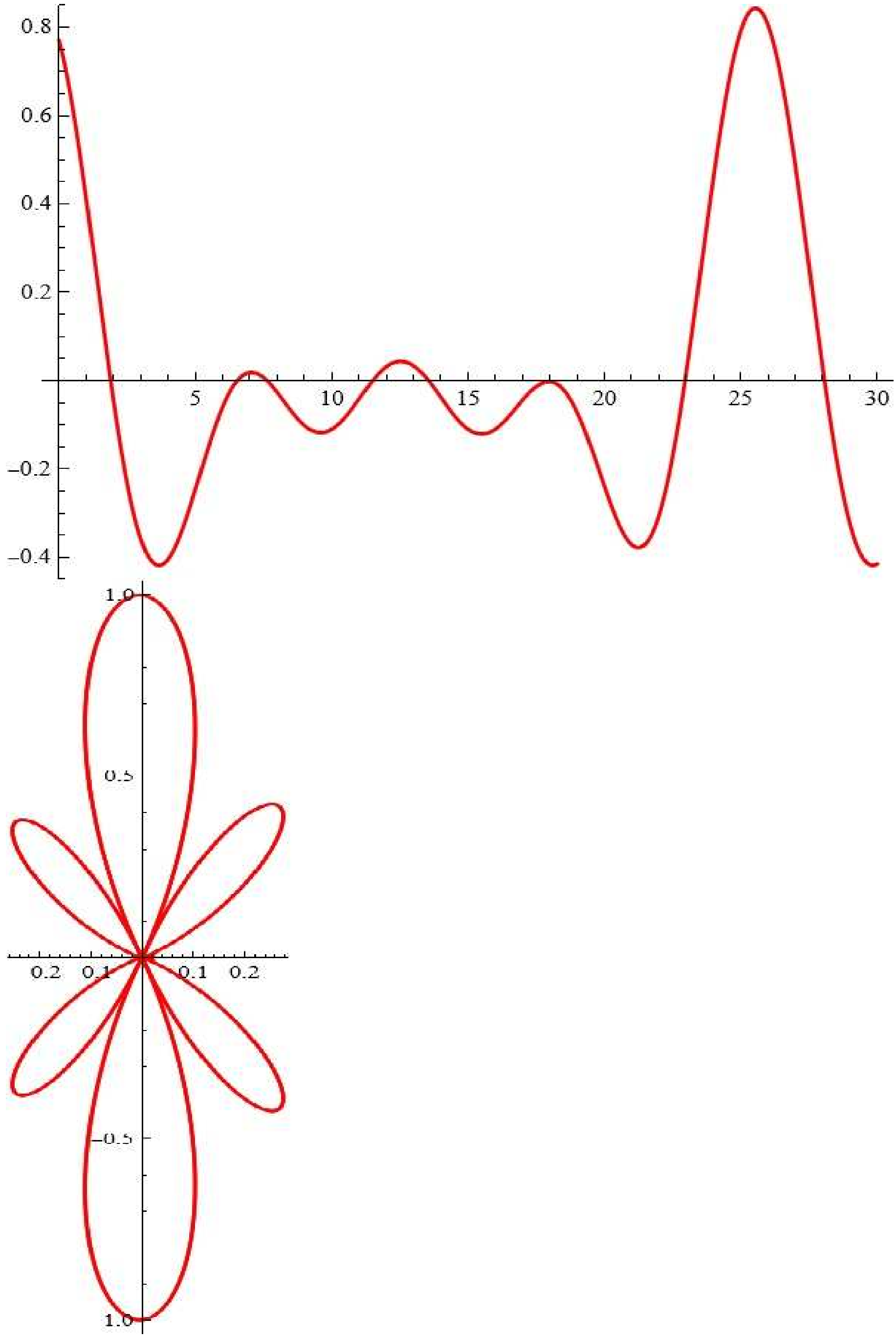

Figure 10: a) the Melnikov function; b) Melnikov antenna factor (Example 4).

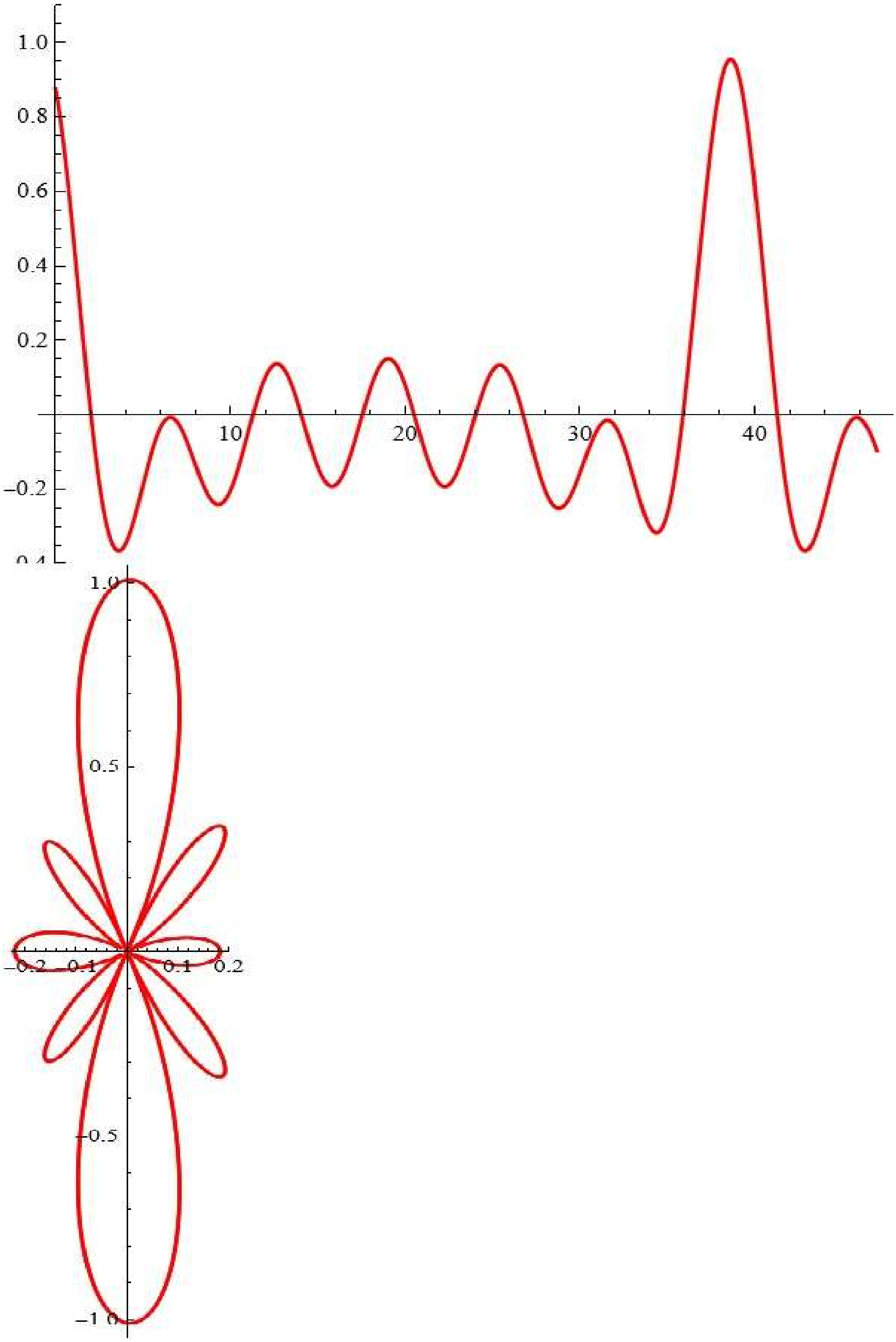

Figure 11: a) the Melnikov function; b) Melnikov antenna factor (Example 5).

Melnikov function and Melnikov antenna factor are depicted in Figure 12.

Example 7. For fixed $N = 10$, $k = 9.1$, $k_1 = -0.5$, $\omega = 0.15$, $a = 1$, $b = 0.31$, $c = 0.001$, $b_1 = 0.31$, $b_2 = 0.31$, $b_3 = 0.19$, $b_4 = 0.12$, $b_5 = 0.12$ $b_6 = 0.12$, $b_7 = 0.12$, $b_8 = 0.12$, $b_9 = 0.31$ Melnikov function and Melnikov antenna factor are depicted in Figure 13.

Of course, this relatively new idea of justification and right to exist is subject to serious research by specialists working in this scientific direction. The issue related to noise minimization (in decibels) also remains open.

4. CONCLUDING REMARKS

1. The existence of subharmonic periodic solutions of system (2) can be proved in the manner detailed in paper [14]. The system (2) is of the form

$$\begin{cases} \dfrac{dx}{dt} & = & f_1(x,y) + \epsilon g_1(x,y,t) \\ \dfrac{dy}{dt} & = & f_2(x,y) + \epsilon g_2(x,y,t) \end{cases}$$

The system ($\epsilon = 0$) has the Hamiltonian $H(x,y) = a^2xy - \frac{1}{2}(x^2+y^2)^2 = \sigma$ with $\sigma \leq \frac{a^4}{8}$. It is known [14] that the family of periodic orbit (in the first quadrant) is of the form

$$q_k(t) = (x_k(t), y_k(t)) : \begin{cases} x_k(t) & = & \frac{a\sqrt{k+1}}{2}\frac{dn(a^2t,k)-ksn(a^2t,k)cn(a^2t,k)}{ksn^2(a^2t,k)+1} \\ y_k(t) & = & \frac{a\sqrt{k+1}}{2}\frac{dn(a^2t,k)+ksn(a^2t,k)cn(a^2t,k)}{ksn^2(a^2t,k)+1} \end{cases}$$

where $k = \frac{\sqrt{a^4-8\sigma}}{a^2}$, and $sn(.,k), cn(.,k), dn(.,k)$ are Jacoby elliptic functions with modulus k. The resonance conditions is $T(k) = \frac{mT}{n}$, in which $T(k)$ is the period of q_k and $T = \frac{2\pi}{\omega}$ is the period of the perturbation. For relatively positive integers m and n the subharmonic Melnikov integral is defined by

$$M^{m/n}(t_0) = \int_{-\frac{mT}{2}}^{\frac{mT}{2}} f(q_k(t)) \wedge g(q_k(t), t+t_0)dt$$

For some details see [14].

It is known [5] that if the subharmonic Melnikov function has a simple zero, and $\frac{dT(k)}{d\sigma(k)} \neq 0$, then for $0 < \epsilon \leq \epsilon(n)$, (2) ($\epsilon = 0$) has a subharmonic orbit of period mT. If $n = 1$, then the result is uniformly valid in $0 \leq \epsilon < \epsilon(1)$. We note that for our new

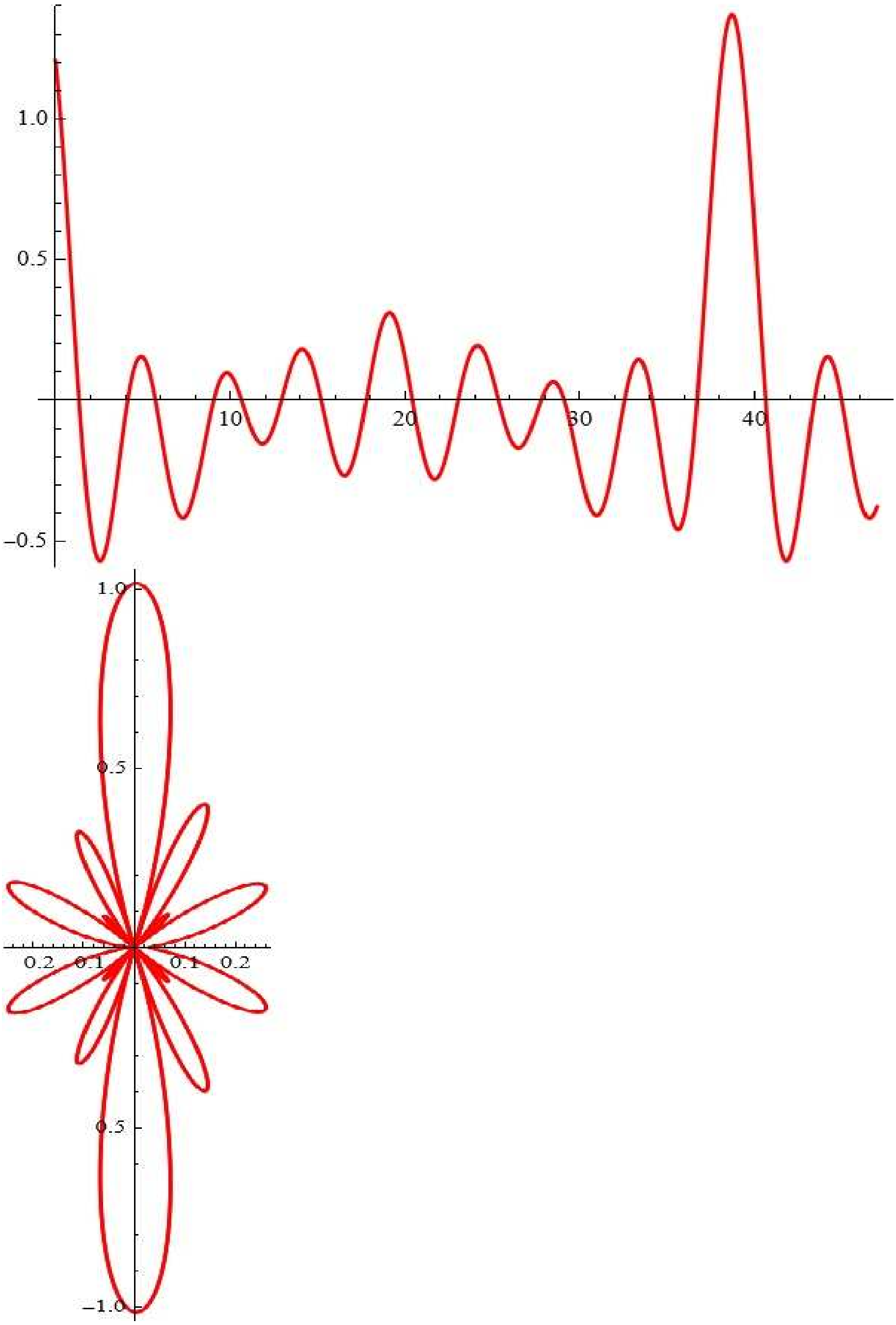

Figure 12: a) the Melnikov function; b) Melnikov antenna factor (Example 6).

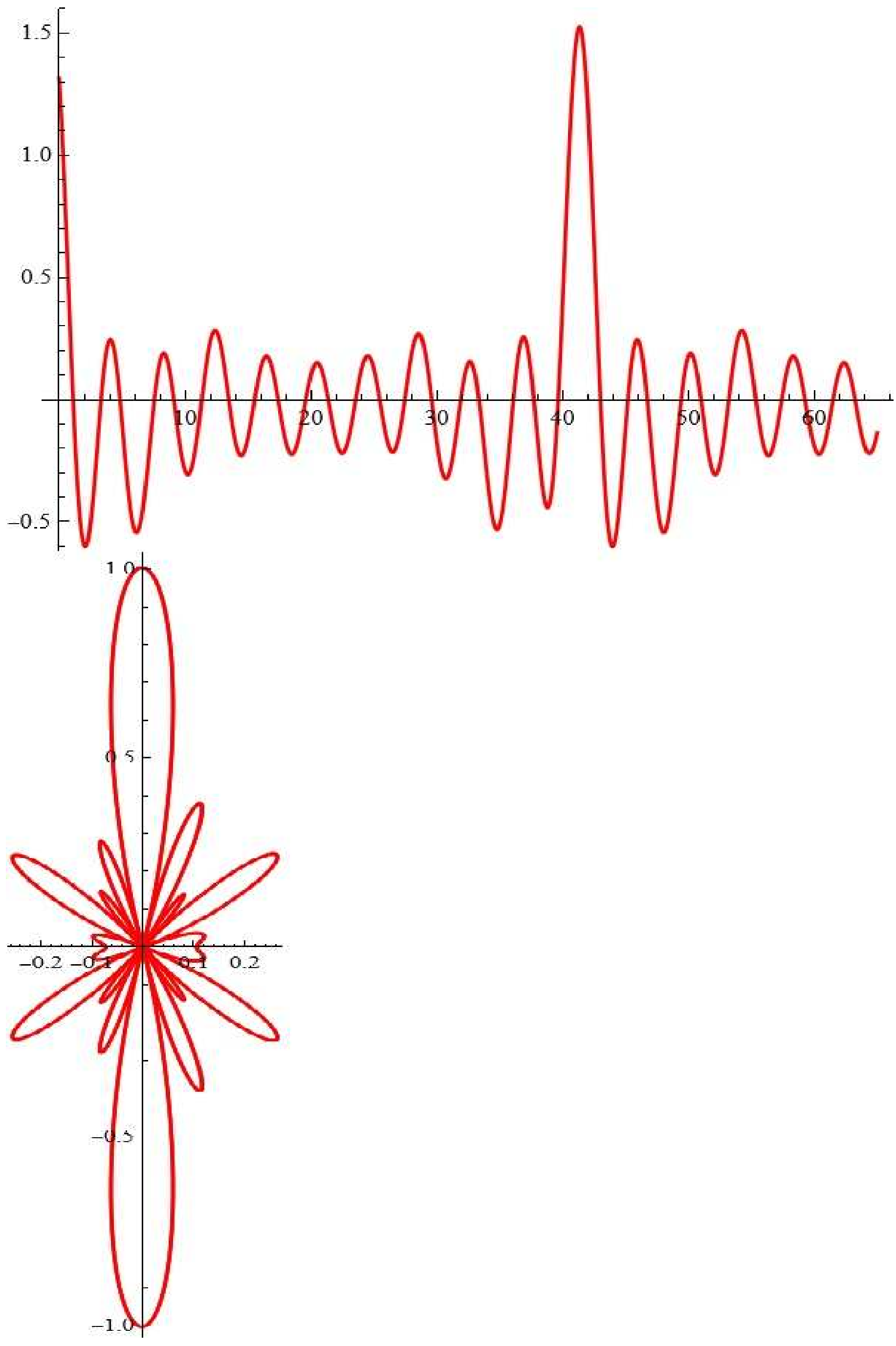

Figure 13: a) the Melnikov function; b) Melnikov antenna factor (Example 7).

model (2) the first subharmonic Melnikov function is:

$$\begin{aligned} M^{m/1}(t_0) = \int_{-\frac{mT}{2}}^{\frac{mT}{2}} (-\tfrac{dy_0(t)}{dt}(b\sin(\omega(t+t_0)) + \sum_{j=2}^{N} b_{j-1}\sin(j\omega(t+t_0))) \\ +a^2 c x_0(t) y_0(t) - 2cx_0^4(t) - 2cx_0^2(t) y_0^2(t))dt \end{aligned} \tag{4}$$

We leave the calculation of the integrals of (4) to the reader.

2. The proposed new extended model contains many free parameters, which makes it attractive for use in the fields of biological applications, reaction kinetics, engineering research etc.

3. In [14], the authors consider the system

$$\begin{cases} \dfrac{dx}{dt} &= ax - \mu m n y^{n-1}(x^n + y^n)^{m-1} \\ \dfrac{dy}{dt} &= -ay + \mu m n x^{n-1}(x^n + y^n)^{m-1} \end{cases} \tag{5}$$

where $m \geq 1$, $n \geq 1$, $mn > 2$, $a > 0$, $\mu > 0$. It is known that the homoclinic orbit in the nonnegative quadrant of the (x, y)-plane is of the form (for $\mu = 1$)

$$x(t) = \frac{a^{\frac{1}{mn-2}} e^{at}}{(1 + e^{an(mn-2)t})^{\frac{m}{mn-2}}}; \quad y(t) = \frac{a^{\frac{1}{mn-2}} e^{a(n-1)t}}{(1 + e^{an(mn-2)t})^{\frac{m}{mn-2}}}; \quad t \in R.$$

By choosing particular values of a, m, n, we obtain particular homoclinic orbits. A look at the following generalized oscillator

$$\begin{cases} \dfrac{dx}{dt} &= ax - \mu m n y^{n-1}(x^n + y^n)^{m-1} + \epsilon \left(cx + \displaystyle\sum_{j=1}^{N} g_j \sin(j\omega t) \right) \\ \dfrac{dy}{dt} &= -ay + \mu m n x^{n-1}(x^n + y^n)^{m-1} \end{cases} \tag{6}$$

is also interest. The dynamics of this new model can be explored using the methodology outlined in this article.

4. The derived results can be used as an integral part of a much more general application for scientific computing – for some details see [15]–[32].

5. Nonstandard numerical methods connected to the investigation of the roots of equation $M(t_0) = 0$ can be found in [33]–[34].

ACKNOWLEDGMENT

The third, fourth, fifth, and sixth authors are supported by the European Union-NextGenerationEU, through the National Plan for Recovery and Resilience of the Republic Bulgaria, project No BG-RP-2.004-0001-C01.

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