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EXISTENCE OF SOLUTIONS TO A CLASS OF SINGULAR SEMILINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract: We prove the existence of weak solutions to a class of singular semilinear second order two-point boundary value problems in a weighted Sobolev's space. The existence result is obtained by applying Schaefer's fixed point theorem. We gave illustrative examples for which our result applies.

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1. Introduction

In applications, singular boundary value problems arise in the fields of boundary layer theory, gas dynamics, combustion, nonlinear optics, etc; see for example [1, 8, 25, 31, 36].

In this paper, we consider the following singular semilinear second order two-point boundary value problems:

$$u''(r) + \frac{a}{r}u'(r) + g(u(r), u'(r)) = h(r), \quad a > 0, \quad r \in (0, 1), \tag{1}$$

$$\lim_{r\to 0^+} u'(r) = 0, \; u(1) = 0. \tag{2}$$

The difficulties in (1) are the presence of the singular term $\frac{a}{r}u'$ and the non-

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linear function $g(u, u')$. Hence, (1) with prescribed boundary conditions are sometimes solved numerically. Over the years, several numerical methods have been developed for solving existing or new problems. For a few of such works, we refer the reader to [7, 15, 17, 19, 24, 27, 28, 35] and the literature cited in them. We remark that the singular equations are either solved with the Dirichlet boundary conditions or mixed Dirichlet/Neumann conditions. In addition, g is assumed to be depend nonlinearly on u.

Examples of existence results for some classes of singular boundary value problems include [3, 10, 12, 13, 16, 20, 21, 22, 33]. Existence of weak or generalized solutions were not considered in the cited works. There are relatively few works on existence of weak solutions for non-singular ordinary differential equations; see for example [9, 18, 32, 34]. There appears to be scarcity of existence results for singular ordinary differential equations in weighted Sobolev's spaces. We are aware of only the current results in [29, 30], in which the nonlinearities are assumed to depend only on the dependent variable u. The class of problems are solved with Dirichlet boundary conditions by the upper and lower solutions method.

Motivated by previous works, we prove the existence of solutions to the semilinear problems (1)–(2) in a subset of the weighted Sobolev's space $H^1[(0, 1), r^a]$. In the current analysis, g is assumed to depend nonlinearly on $u(r)$ and $u'(r)$; and we impose the Dirichlet/Neumann boundary conditions. We assume $h \in L^2[(0, 1), r^a]$ and a nonsmooth nonlinearity $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which satisfies the following conditions:

H $\quad |g(s, t) - g(v, w)| \leq L(|s - v| + |t - w|)$ (Lipschitz continuity),

for some constant $L > 0$. Our existence result is based on an application of the Schaefer's fixed point theorem. For information on weighted Sobolev spaces we refer the reader to [4] and [23].

The remaining part of this paper is arranged as follows: In Section 2, the weighted Sobolev's spaces used in this paper are defined. Further, we give an equivalent equation to (1) and the definition of a weak solution to the problems (1)–(2). We prove the existence of solution to some auxiliary linear problems in Section 3. In Section 4, we establish the existence of weak solutions to the problems (1)–(2). We finally give an illustrative example for which our result applies in Section 5.

2. Preliminaries

Definition 1. Let

$$L^2[(0,1),r^a] := \{w : (0,1) \to \mathbb{R} \text{ such that } \|w\|_{L^2[(0,1),r^a]} < \infty\},$$

where $\|w\|_{L^2[(0,1),r^a]} = \sqrt{\int_0^1 r^a w^2 dr}$.

Let

$$H^1[(0,1),r^a] := \{w : (0,1) \to \mathbb{R} \text{ such that } \|w\|_{H^1[(0,1),r^a]} < \infty\},$$

where $\|w\|_{H^1[(0,1),r^a]} = \sqrt{\int_0^1 r^a w^2 dr + \int_0^1 r^a w'^2 dr}$.

Let

$$X := \{w \in H^1[(0,1),r^a] \mid \lim_{r\to 0^+} u'(r) = 0,\ u(1) = 0\}$$

.

We will use the following equivalent equation to (1.1) in our analysis:

$$-(r^a u')' = r^a[g(u,u') - h] \tag{3}$$

Definition 2. We say that $u \in X$ is a weak solution of (1)–(2) provided

$$\int_0^1 r^a u'\zeta' dr = \int_0^1 r^a g(u,u')\zeta dr - \int_0^1 r^a h\zeta dr, \tag{4}$$

for each $\zeta \in X$.

Definition 3. Let X be a Banach space, and $A : X \to X$ a nonlinear mapping. A is called compact provided for each bounded sequence $\{u_k\}_{k=1}^{\infty}$ the sequence $\{A[u_k]\}_{k=1}^{\infty}$ is precompact; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{A[u_{k_j}]\}_{j=1}^{\infty}$ converges in X (see [11]).

The following theorems are applied in this paper:

Theorem 4. *(Bolzano–Weierstrass). Every bounded sequence of real numbers has a convergent subsequence (see [26]).*

Theorem 5. *(Schaefer's Fixed Point Theorem). Let X be a Banach space and*

$$A : X \to X$$

a continuous and compact mapping. Suppose further that the set

$$\{u \in X \mid u = \tau A[u] \ for\ some\ 0 \le \tau \le 1\}$$

is bounded. Then A has a fixed point (see [11]).

Remark. Notice the advantage of Schaefer's fixed point theorem over Schauder's fixed point theorem for applications: We need not identify any convex, compact set [11].

3. Auxiliary linear problems

Consider the linear boundary value problems

$$Lu := -(r^a u')' + \mu r^a u = \mu r^a s + r^a[g(s,s') - h], \quad a > 0, \quad r \in (0,1) \tag{5}$$
$$\lim_{r\to 0^+} u'(r) = 0, \; u(1) = 0, \tag{6}$$

where the constant $\mu > 0$, $s \in X$ is fixed, $h \in L^2[(0,1), r^a]$ and $g(s,s')$ satisfies the Lipschitz condition H.

Theorem 6. *(A priori estimates). Let u be a solution of* (5)–(6)*. Then $u \in X$ and we have the estimate*

$$\|u\|_X \le C\sqrt{\|s\|_X^2 + \|h\|^2_{L^2[(0,1),r^a]} + 1} < \infty, \tag{7}$$

for some constant $C > 0$.

Proof. We split the proof in two steps.
STEP 1. Multiply (5) by u, integrate by parts and apply (6) to get

$$\int_0^1 r^a u'^2 dr + \mu \int_0^1 r^a u^2 dr = \mu \int_0^1 r^a us dr + \int_0^1 r^a u g(s,s') dr - \int_0^1 r^a uh dr$$
$$\le \mu \left(\int_0^1 r^a u^2 dr\right)^{\frac12} \left(\int_0^1 r^a s^2 dr\right)^{\frac12} + \left(\int_0^1 r^a u^2 dr\right)^{\frac12} \left(\int_0^1 r^a h^2 dr\right)^{\frac12} +$$
$$\left(\int_0^1 r^a u^2 dr\right)^{\frac12} \left(\int_0^1 r^a |g(s,s')|^2 dr\right)^{\frac12}, \text{ (by Hölder's inequality)},$$
$$\le \left(\epsilon \int_0^1 r^a u^2 dr + \frac{1}{4\epsilon}\mu^2 \int_0^1 r^a s^2 dr\right) + \left(\epsilon \int_0^1 r^a u^2 dr + \frac{1}{4\epsilon}\int_0^1 r^a h^2 dr\right) +$$
$$\left(\epsilon \int_0^1 r^a u^2 dr + \frac{1}{4\epsilon}\int_0^1 r^a |g(s,s')|^2 dr\right) \quad \text{(by Cauchy's inequality with } \epsilon\text{)},$$
$$\le 3\epsilon \int_0^1 r^a u^2 dr + \frac{1}{4\epsilon}\left(\mu^2 \int_0^1 r^a s^2 dr + \int_0^1 r^a |g(s,s')|^2 dr + \int_0^1 r^a h^2 dr\right) \tag{8}$$

STEP 2. We now estimate the second term in the bracket on the right side. Using the condition H, we have

$$\int_0^1 r^a |g(s,s') - g(0,0)|^2 dr \le L^2 \int_0^1 r^a \left(|s| + |s'|\right)^2 dr \text{ or}$$

$$\int_0^1 r^a|g(s,s')|^2dr \leq -\int_0^1 r^a|g(0,0)|^2dr + 2\int_0^1 r^a g(s,s')g(0,0)dr + 2L^2\|s\|_X^2$$
$$\leq \int_0^1 r^a|g(0,0)|^2dr + \int_0^1 [r^{\frac{a}{2}}g(s,s')][2r^{\frac{a}{2}}g(0,0)]dr + 2L^2\|s\|_X^2$$
$$\leq \int_0^1 r^a|g(0,0)|^2dr + \frac{1}{2}\int_0^1 [r^{\frac{a}{2}}g(s,s')]^2dr + \frac{1}{2}[2r^{\frac{a}{2}}g(0,0)]^2dr + 2L^2\|s\|_X^2$$
$$\leq -\int_0^1 r^a|g(0,0)|^2dr + \frac{1}{2}\int_0^1 r^a|g(s,s')|^2dr + 2\int_0^1 r^a|g(0,0)|^2dr + 2L^2\|s\|_X^2,$$

(by Young's inequality). (9)

(9) can be simplified to deduce

$$\int_0^1 r^a|g(s,s')|^2dr \leq C\left(\|s\|_X + 1\right)^2 \tag{10}$$

Using (10) and choosing $\epsilon > 0$ sufficiently small in (8) and simplifying, we deduce (7). □

Definition 7. (i) The *bilinear form* $B[.,.]$ associated with the elliptic operator L defined by (5) is

$$B[u,\zeta] := \int_0^1 r^a u'\zeta' dr + \mu\int_0^1 r^a u\zeta dr,$$

for $u,\zeta \in X$.

(ii) $u \in X$ is called a *weak solution* of the boundary value problems (5)–(6) provided

$$B[u,\zeta] = (\mu r^a s + r^a[g(s,s') - h],\zeta),$$

for all $\zeta \in X$, where $(.,.)$ denotes the inner product in $L^2(0,1)$.

Theorem 8. *The bilinear form $B[u,\zeta]$ satisfies the hypotheses of the Lax-Milgram theorem. That is, there exists constants α,β such that*

(i) $B[u,\zeta] \leq \alpha\|u\|_X\|\zeta\|_X$,

(ii) $\beta\|u\|_X^2 \leq B[u,u]$,

for all $u,\zeta \in X$.

Proof. We compute

$$|B[u,\zeta]| = |\int_0^1 r^a u'\zeta' dr + \mu\int_0^1 r^a u\zeta dr| \leq \left(\int_0^1 r^a u'^2 dr\right)^{\frac{1}{2}}\left(\int_0^1 r^a \zeta'^2 dr\right)^{\frac{1}{2}}$$
$$+\left(\int_0^1 r^a u^2 dr\right)^{\frac{1}{2}}\left(\int_0^1 r^a \zeta^2 dr\right)^{\frac{1}{2}} \text{ (by Hölder's inequality)}$$
$$\leq \alpha\|u\|_X\|\zeta\|_X \tag{11}$$

for appropriate constant $\alpha > 0$. This proves (i).

We now prove (ii). We check that

$$\beta\|u\|_X^2 = \beta\left(\int_0^1 r^a u'^2 dr + \int_0^1 r^a u^2 dr\right) \le \int_0^1 r^a u'^2 dr + \mu\int_0^1 r^a u^2 dr = B[u,u], \quad (12)$$

for $\beta := \min\{1, \mu\} > 0$. □

Theorem 9. *There exist weak solutions of the linear boundary value problems* (5)–(6).

Proof. (10) and the hypothesis on h imply that $g(s,s') - h \in L^2[(0,1), r^a]$. Let $\langle ., .\rangle$ denote the pairing of X with its dual. For fixed $g(s,s') - h$, set $\langle \mu r^a s + r^a[g(s,s') - h], \zeta\rangle := (\mu r^a s + r^a[g(s,s') - h], \zeta)_{L^2(0,1)}$ for all $\zeta \in X$. This is a bounded linear functional on $L^2(0,1)$ and thus on X. Lax-Milgram theorem (see for example [11]) can be applied to find a unique function $v \in X$ satisfying

$$B[u,\zeta] = \langle \mu r^a s + r^a[g(s,s') - h], \zeta\rangle$$

for all $\zeta \in X$. u is consequently the unique weak solutions of the problems (5)–(6). □

4. Main result

Theorem 10. *There exist weak solutions of the boundary value problems* (1)–(2).

Proof. We split the proof in six steps.
STEP 1. A fixed point argument to (1)–(2) is

$$-(r^a w')' + \mu r^a w = \mu r^a u + r^a[g(u,u') - h], \ a > 0, \ r \in (0,1) \quad (13)$$

$$\lim_{x\to 0^+} w'(r) = 0, \ w(1) = 0. \quad (14)$$

Define a mapping

$$\tau \ : \ X \to X.$$

Write $\tau[u] = w$ whenever w is derived from u via (13)–(14). We claim that τ is a continuous and compact mapping. We will proof our claim in the next two steps.
STEP 2. Choose $u, \tilde{u} \in X$, and define $\tau[u] = w, \ \tau[\tilde{u}] = \tilde{w}$. For two solutions $w, \tilde{w} \in X$ of (13)–(14), we have

$$-[r^a(w' - \tilde{w}']' + \mu r^a(w - \tilde{w}) = \mu r^a(u - \tilde{u}) + r^a[g(u,u') - g(\tilde{u},\tilde{u}')], \quad (15)$$

$$\lim_{r\to 0^+}(w-\tilde{w})'(r)=0,\ (w-\tilde{w})(1)=0. \tag{16}$$

Using (15)–(16), we get an analogous estimate to (7) of Theorem 6, viz:

$$\begin{aligned}
\|w-\tilde{w}\|_X &\le C_1\sqrt{\int_0^1 r^a(u-\tilde{u})^2dr+\int_0^1 r^a|g(u,u')-g(\tilde{u},\tilde{u}')|^2dr}\\
&\le C_1\sqrt{\int_0^1 r^a(u-\tilde{u})^2dr+2L^2\int_0^1 r^a(|u-\tilde{u}|^2+|u'-\tilde{u}'|^2)dr},\ (\text{using } H)\\
&\le C\|u-\tilde{u}\|_X,\ (\text{for some constant } C>0).
\end{aligned} \tag{17}$$

Using (17) and the definition of the mapping τ, we have

$$\|\tau[u]-\tau[\tilde{u}]\|_X=\|w-\tilde{w}\|_X\le C\|u-\tilde{u}\|_X, \tag{18}$$

so that the mapping τ is Lipschitz continuous, and hence continuous.
STEP 3. Let $\{u_k\}_{k=1}^{\infty}$ be a bounded sequence in X. It has a convergent subsequence, say $\{u_{k_j}\}_{j=1}^{\infty}$, by Bolzano-Weierstrass theorem (Theorem 4). Define

$$u:=\lim_{k_j\to\infty}u_{k_j}. \tag{19}$$

Using (18)–(19), we deduce

$$\lim_{k_j\to\infty}\|\tau[u_{k_j}]-\tau[u]\|_X\le\lim_{k_j\to\infty}\|u_{k_j}-u\|_X=0. \tag{20}$$

Hence, $\tau[u_{k_j}]\to\tau[u]$ in X. Therefore, τ is compact.
STEP 4. Let a set K be defined as $K:=\{v\in X\ :\ v=\gamma\tau[v] \text{ for some } 0\le\gamma\le 1\}$. We will show that the set K is bounded. Let $u\in K$. Then $u=\gamma\tau[u]$ for some $\gamma\in[0,1]$. We therefore have that $\frac{u}{\gamma}=\tau[u]$. By the definition of the mapping τ, $w=\frac{u}{\gamma}$ is the solution of the problem

$$-\left(r^a\left(\frac{u}{\gamma}\right)'\right)'+\mu r^a\left(\frac{u}{\gamma}\right)=\mu r^a u+r^a[g(u,u')-h],\ a>0,\ r\in(0,1) \tag{21}$$

$$\lim_{r\to 0^+}\left(\frac{u'(r)}{\gamma}\right)(r)=0,\ \left(\frac{u}{\gamma}\right)(1)=0. \tag{22}$$

Now, (21)–(22) are equivalent to

$$-(r^au')'+\mu r^au=\ \gamma\mu r^au+\gamma r^a[g(u,u')-h],\quad a>0,\quad r\in(0,1) \tag{23}$$

$$\lim_{r\to 0^+}u'(r)=0,\ u(1)=\gamma A. \tag{24}$$

Multiplying (23) by u, integrating by parts and applying (24), we deduce

$$\int_0^1 r^au'^2dr+\mu\int_0^1 r^au^2dr=\gamma\mu\int_0^1 r^au^2dr+\gamma\int_0^1 r^au[g(u,u')-h]dr$$

$$\leq \gamma \left(\mu \int_0^1 r^a u^2 dr + \int_0^1 r^a u^2 dr + \frac{1}{2} \int_0^1 r^a |g(u,u')|^2 dr + \frac{1}{2} \int_0^1 r^a h^2 dr \right)$$

(using Young's inequality and simplifying)

$$\leq \gamma \left((\mu + 1) \int_0^1 r^a u^2 dr + C_1 \left(\|u\|_X + 1 \right)^2 + \frac{1}{2} \int_0^1 r^a h^2 dr \right), \tag{25}$$

using (10), for some constant $C_1 > 0$. Choosing $\gamma \in [0,1]$ sufficiently small and simplifying, we deduce that

$$\|u\|_X \leq C\sqrt{1 + \|h\|^2_{L^2[(0,1),r^a]}} < \infty, \tag{26}$$

for some constant $C > 0$. (26) implies that the set K is bounded, since u was arbitrarily chosen.

Since the mapping τ is continuous and compact; and the set K is bounded, by Schaefer's fixed point theorem (Theorem 5), the mapping τ has a fixed point in X.

STEP 5. Write $u_0 = 0$. For $k = 0, 1, 2, \ldots$, inductively define $u_{k+1} \in X$ to be the unique weak solutions of the linear boundary value problems

$$-(r^a u'_{k+1})' + \mu r^a u_{k+1} = \mu r^a u_k + r^a [g(u_k, u'_k) - h], \tag{27}$$

$$\lim_{r \to 0^+} u'_{k+1}(0) = 0, \; u_{k+1}(1) = 0, \quad a > 0, \; r \in (0,1). \tag{28}$$

Notice that our definition of $u_{k+1} \in X$ as the unique weak solutions of the problems (27)–(28) is justified by Theorem 9. Thus, by the definition of the mapping τ, we have for $k = 0, 1, 2, \ldots$:

$$u_{k+1} = \tau[u_k].$$

Since τ has a fixed point in X, there exists $u \in X$ such that

$$\lim_{k \to \infty} u_{k+1} = \lim_{k \to \infty} \tau[u_k] = \tau[u] = u. \tag{29}$$

Furthermore, we deduce from (10) that

$$\|g(u_k, u'_k)\|_{L^2[(0,1),r^a]} \leq C \left(\|u_k\|_X + 1 \right) \tag{30}$$

for some constant $C > 0$. We use (29) to obtain the limit on the right side of (30) to deduce

$$\sup_k \|g(u_k, u'_k)\|_{L^2[(0,1),r^a]} < \infty. \tag{31}$$

(31) implies the existence of a subsequence $\{g(u_{k_j}, u'_{k_j})\}_{j=1}^{\infty}$ converging weekly in $L^2[(0,1), r^a]$ to $g(u, u')$ in $L^2[(0,1), r^a]$.

STEP 6. We now verify that u is a weak solution of (1)-(2). For brevity, we take the subsequence of the last step as $\{g(u_k')\}_{k=1}^{\infty}$. Fix $\zeta \in X$. Multiply (27) by ζ, integrate by parts and apply (28) to deduce

$$\int_0^1 r^a u_{k+1}'\zeta' dr + \mu \int_0^1 r^a u_{k+1}\zeta dr = \mu \int_0^1 r^a u_k \zeta dr + \int_0^1 r^a g(u_k')\zeta dr - \int_0^1 r^a h\zeta dr. \quad (32)$$

Using (29) and letting $k \to \infty$ in (32), we obtain

$$\int_0^1 r^a u'\zeta' dr + \mu \int_0^1 r^a u\zeta dr = \mu \int_0^1 r^a u\zeta dr + \int_0^1 r^a g(u, u')\zeta dr - \int_0^1 r^a h\zeta dr.$$

from whence canceling the terms in μ, we obtain (4) as desired. $\square$

5. Illustrative examples

Example 11. Consider the problem

$$u''(r) + \frac{a}{r}u'(r) = -1, \quad a > 0, \quad r \in (0, 1), \quad (33)$$

$$\lim_{r\to 0^+} u'(r) = 0, \ u(1) = 0. \quad (34)$$

$g(u, u') \equiv 0$ is trivially Lipschitz continuous and $h(r) = -1 \in L^2[(0, 1), r^a]$. This problem admits a classical solution

$$u = \frac{1 - r^2}{2(a + 1)},$$

which is trivially in X.

Example 12. Notice that for the singular boundary value problem

$$u''(r) + \frac{1}{r}u'(r) + u'(r) = -r, \quad r \in (0, 1), \quad (35)$$

$$\lim_{r\to 0^+} u'(r) = 0, \ u(1) = 0, \quad (36)$$

$a = 1$, $g(u, u') = u'$ is Lipschitz continuous and $h(r) = -r \in L^2[(0, 1), r]$. By Theorem 10, there exists a solution

$$u \in X_1 := \{u \in H^1[(0, 1), r] \mid \lim_{r\to 0^+} u'(r) = 0, \ u(1) = 0\}$$

to the boundary value problem (35)–(36). Indeed this problem admits a solution

$$u(r) = \int_r^1 \frac{\int_0^\sigma s^2 e^s ds}{\sigma e^\sigma} d\sigma. \quad (37)$$

It is not difficult to show that $u \in X_1$.

Example 13. Consider the singular two-point boundary value problem:

$$u''(r) + \frac{a}{r}u'(r) + \frac{1}{1+|u+u'|} = r^{-\frac{a}{4}}, \quad a > 0, \quad r \in (0,1), \tag{38}$$

$$\lim_{r\to 0^+} u'(r) = 0, \ u(1) = 0. \tag{39}$$

Nonsmooth nonlinearity $g(u,u') := \frac{1}{1+|u+u'|}$ satisfies the condition H, and $h := r^{-\frac{a}{4}} \in L^2[(0,1), r^a]$. Theorem 10 guarantees the existence of a solution $u \in X$ of the singular two-point boundary value problem (38)–(39).

6. Conclusion

The existence result of Section 4, serves to reduce the long and tedious analyses expected of all problems belonging to the class of problems represented by (1)–(2). Consequently, existence of solutions to the 3 problems in Section 5 are easily inferred by applying Theorem 10. Notice that the existence deductions on the 3 problems were made on one page!

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