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ON STABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN FOURTH ORDER DELAY DIFFERENTIAL EQUATION

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Abstract: In this paper, sufficient criteria which guarantee the existence of uniform asymptotic stability and boundedness of solution of a scalar real fourth-order delay differential equation were established with the aid of a suitable Lyapunov function. With the Lyapunov function, conditions on the nonlinear terms to guarantee stability and boundedness of the solution and its derivatives were given. Our results include and improve on the existing results in the literature.

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1. Introduction

In this paper, we consider the fourth order delay differential equation

$$x^{(iv)} + a \dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}(t-\tau)) + h(x(t-\tau)) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \qquad (1.1)$$

where functions f, g, h and p are continuous and depend (at most) only on

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the arguments displayed explicitly; a being a constant and $\tau > 0$ a fixed delay. Here and elsewhere, all the solutions considered and all the functions which appear are supposed real. The dots indicate differentiation with respect to t. When $\tau = 0$ the above equation reduces to an ordinary nonlinear fourth order differential equation

$$x^{(iv)} + a\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}(t)) + h(x(t)) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}). \qquad (1.2)$$

Various forms of (1.2) have received great attention by researchers (see for instance [1],[6],[10]-[11], [15], [18], [26]-[27], [28]-[29], [31], [36] and the references therein). In all the cited references above the use of the Lyapunov second method was used extensively by the researchers to discuss the qualitative properties of various form of nonlinear fourth order differential equations without delay. Some of these results have been summarized in [23].

The Lyapunov second method had also been found useful and applicable to study the qualitative properties of the differential equation with delay (see [2], [3]-[4], [8], [20], [21]-[22], [24], [25], [30], [32]-[35]).

In [3], a study on second order equation was carried out and the author constructed a Lyapunov function which was later converted to a Lyapunov functional to discuss the qualitative properties of solutions of the second order delay differential equation considered. Hale also in [14] used Lyapunov functions to give sufficient conditions for stability and boundedness of a first order and second order delay differential equations while Razumikhin in [21] gave a fundamental procedure where a nonlinear differential equation with delay could be discussed as approximation to linear differential equations.

In [20], the author used the Razumkhin-type theorem to deduce sufficient conditions that guarantee the uniform asymptotic stability and boundedness of solution of a scalar real fourth-order delay differential equation considered.

In [24], the author constructed a Lyapunov functional and used it to give group of conditions to ensure that the zero solution of

$$x^{(iv)} + a_1\dddot{x} + a_2\ddot{x} + \phi(\dot{x}(t)) + f(x(t)) = 0.$$

is globally asymptotically stable when the delay τ is suitably small.

In [19], the author adapted [3] and [21] and use a suitable complete Lyapunov function to establish criteria which guarantee existence of unique solution that is bounded together with its derivatives on the real line, globally stable and periodic under explicit conditions on the nonlinear terms of the third order equation considered.

Our aim in this work is to adopt the approach in [3] and [21] to extend the result in [18] to the equation (1.1) and give sufficient criteria which guarantee the existence of uniform asymptotic stability and boundedness of the solution with their derivatives on the real line.

The rest of this paper is organized as follows. In Section 2 we gave the basic assumptions in formulating our results alongside with our main results. Section 3 is devoted to some preliminary results and in Section 4 the proofs of the main theorems are given.

We shall investigate equation (1.1) in the equivalent form given as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - f(x,y)z - g(y) - h(x) + M(t), \end{aligned} \tag{1.3}$$

where

$$M(t) = \int_{-\tau}^{0} [g'(y(t+\theta))z(t+\theta) + h'(x(t+\theta))y(t+\theta)]d\theta + p(t)$$

We shall for completeness sake like to refer the reader to [5], [7], [9],[12], [13], [16]-[17] and [37] for terminologies, techniques and standard results.

2. Formulation of Results

Now let the functions $f, g,\ h$ and p be continuous and the following conditions hold:

(i)
$$I_0 = [\delta, \Delta],\ \delta, \Delta > 0,\ \text{and } I_0 \text{ is the Routh Hurwitz interval.}$$

(ii)
$$\frac{h(x) - h(0)}{x} = H_0 \leq \alpha \in I_0 \quad x \neq 0\ ;$$

(iii)
$$\frac{g(y) - g(0)}{y} = G_0 \leq \beta \in I_0, \quad y \neq 0$$

(iv)
$$|f(x,y)| \leq \rho \text{ positive constant}$$

(v)
$$h(0) = g(0) = 0.$$

where α, β are all positive.

Next,we state our main results.

Theorem 2.1. *Suppose that conditions (i)-(v) are satisfied with* $p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \equiv 0$, *then the trivial solution of the equation* (1.1) *is uniformly asymptotically stable.*

Theorem 2.2. *In addition to conditions (i)-(v) being satisfied, suppose that*

(vi)

$$p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \equiv p(t) \quad \text{and} \quad |p(t)| \leq \kappa,$$

for all $t \geq 0$, *then there exists a constant* σ, $(0 < \sigma < \infty)$ *depending only on the constants* α, β, ρ *and* δ *such that every solution of* (1.1) *satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq e^{-\sigma t} \left\{ A_1 + A_2 \int_{t_0}^{t} |p(\tau)| \, e^{\frac{1}{2}\sigma\tau} d\tau \right\}^2,$$

for all $t \geq t_0$, *where the constant* $A_1 > 0$, *depends on* α, β, ρ *and* δ *as well as on* $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dddot{x}(t_0)$; *and the constant* $A_2 > 0$ *depends on* α, β, ρ *and* δ.

Theorem 2.3. *Suppose the conditions of the Theorem 2.2 are satisfied with condition (vi) replaced with*

(vii)

$$|p(t, x, \dot{x}, \ddot{x}, \dddot{x})| \leq (|x| + |y| + |z| + |w|)\theta(t),$$

where $\theta(t)$ *is a non negative and continuous function of* t *such that* $\int_0^t \theta(s) ds \leq \kappa < \infty$ *is satisfied with a positive constant* κ. *Then, there exists a constant* K_0 *which depends on* κ, K_1, K_2 *and* t_0 *such that every solution* $x(t)$ *of the equation* (1.1) *satisfies*

$$|x(t)| \leq K_0, \qquad |\dot{x}(t)| \leq K_0 \qquad |\ddot{x}(t)| \leq K_0, \qquad \left|\dddot{x}(t)\right| \leq K_0$$

for sufficiently large t.

Remark: We wish to remark here that while the Theorem 2.1 is on the uniform asymptotic stability of the trivial solution, the Theorems 2.2 and 2.3 deal with the boundedness and ultimate boundedness of the solutions respectively.

Notations. Throughout this paper $K, K_0, K_1, \ldots K_{12}$ will denote finite positive constants whose magnitudes depend only on the functions f, g, h and

p as well as constants α, β, ρ and δ but are independent of solutions of the equation (1.1). $K_i'^{}s$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

3. Preliminary Results

We shall use as a tool to prove our main results a Lypunov function $V(t; x, y, z, w)$ defined by

$$V(t; x, y, z, w) = V_1(x, y, z, w) + V_2(t; x, y, z, w), \qquad (3.1)$$

with

$$V_1(x, y, z, w) = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Gxw + 2Hyz + 2Iyw + 2Jzw, \qquad (3.2)$$

where:

$$\begin{aligned}
A =& \frac{a\delta}{\Delta} \left\{(b + d)(c^2 + d^2)[d(1 - ad) - c] + d^3[a(b^2 + d^2) + L]\right\}, \\
B =& \frac{\delta}{\Delta} \left\{dL(abd + c) + a(b^2 + d^2)[b(d - c) + cd] \right. \\
& \left. + [d(1 - ad) - c][ad(b^2 + c^2) - cd^2(b + 1) + a^2bc]\right\}, \\
C =& \frac{\delta}{\Delta} \left\{a(b^2 + d^2)[d(1 - ad + a^2c + d) - c] \right. \\
& \left. + d[c(a^2 + b^2) - ab][d(1 - ad) - c] + dL(a^2c + d)\right\}, \\
D =& \frac{cd\delta}{\Delta} \left\{L + ab^2 + (d - c) + ab[(1 - ad) - c]\right\}, \\
E =& \frac{ac\delta}{\Delta} \left\{d^2L + (b^2 + d^2)(d - c)\right\}, \\
F =& \frac{cd\delta}{bd\Delta} \left\{d^2L + ad^2(b^2 + d^2) + [b(a^2 + d^2) + d^2][ab^2d^2[d(1 - ad) - c]]\right\}, \\
G =& \frac{abc[d(1 - ad - c)]\delta}{\Delta}, \\
H =& \frac{abcd\delta}{\Delta} \left\{a(b^2 + d^2) + L\right\}, \\
I =& \frac{a\delta}{\Delta} \left\{d^2L + bd[d(1 - ad) - c] + (b^2 + d^2)(d - c)\right\}, \\
J =& \frac{acd\delta}{\Delta} \left\{ab^2 + d - c + L\right\}, \\
\Delta =& abcd[d(1 - ad) - c],
\end{aligned}$$

$$L = b[ad + c[c(b+1) - c]], \tag{3.3}$$

and

$$V_2(t; x, y, z, w) = \frac{\gamma}{2\tau} \int_{-\tau}^{0} \left\{ \int_{\theta_1}^{0} [x^2(t+\theta) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)] d\theta \right\} d\theta_1, \tag{3.3}$$

with $a, b, c, d, \alpha, \beta, \rho, \Delta, \gamma, \delta$ and τ all positive for all x, y, z, w.

The following lemmas are needed in the proofs of Theorems 2.1, 2.2 and 2.3.

Lemma 3.1. *Subject to the assumptions of Theorem 2.1 there exist positive constants* $K_i = K_i(a, b, c, d, \delta), i = 1, 2$ *such that*

$$K_1(x^2 + y^2 + z^2 + w^2) \leq V(t; x, y, z, w) \leq K_2(x^2 + y^2 + z^2 + w^2). \tag{3.4}$$

Proof. First, it is clear from the equation (3.1) that clearly $V(0, 0, 0, 0) \equiv 0$. By re-arranging equation (3.2) we have,

$$\begin{aligned}
2V_1(x, y, z, w) = &\left(\frac{\delta}{\Delta}\right) \left\{ a[d(1-ad)]\{b(cx + dy + w)^2 + d^2(y + b^3 d^2 x)^2 \right. \\
&\left. + b^2 d(y + a^2 bdx)^2 + acd(z + \frac{b^2 d^3}{a} x)^2 \right\} + dL \left\{ (z + acx)^2 \right. \\
&\left. + ac^2(z + \frac{1}{a} w)^2 + c(y + \frac{ad}{c} w)^2 + ad^2(x + \frac{c}{d} y)^2 + abd(y + \frac{c}{d} z)^2 \right\} \\
&+ ad(b^2 + d^2) \left\{ ad^2(x + \frac{c(d-c)}{ad^3} y)^2 + a^2 c(z + \frac{d}{a} x)^2 \right. \\
&\left. + \frac{c}{a(b^2 + d^2)} (w + az)^2 + b(d-c)(y + \frac{w}{b})^2 + c(y + abz)^2 \right\} \\
&+ \left\{ [d(1-ad) - c](ad(c^2 + d^2) + abd^2) - \frac{cd^3}{a}(b^2 + d^2) - b^4 cd^3 \right. \\
&\left. - a^5 b^4 d^3 - ab^6 d^4 - a^2 c^2 d^2 L \right\} x^2 \\
&+ \left\{ [d(1-ad) - c][ad(b^2 + c^2) - cd^2(b+1) + a^2 bc - abd^2] - ac^2 dL \right. \\
&\left. - \frac{c^2(d-c)^2}{d^3} \right\} y^2 + \left\{ ad^2(b^2 + d^2) + d(b^2 c - ab)[d(1-ad) - c] \right. \\
&\left. - a^3 b^2 cd(b^2 + d^2) - abc^2 L - a^2 cd[ab^2 + (d-c)] \right\} z^2
\end{aligned}$$

$$+\left\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\right\}w^2\}, \quad (3.5)$$

simplifying further yields,

$$\begin{aligned}2V_1(x,y,z,w) \geq \left(\frac{\delta}{\Delta}\right)&\{\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)\\ &-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\}x^2\\ &+\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL\\ &-\frac{c^2(d-c)^2}{d^3}\}y^2+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]\\ &-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\}z^2\\ &+\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\}w^2\}\\ &\geq K_1(x^2+y^2+z^2+w^2),\end{aligned} \quad (3.6)$$

where

$$\begin{aligned}K_1=\frac{\delta}{\Delta}\min\{&|[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)-b^4cd^3\\ &-a^5b^4d^3-ab^6d^4\\ &-a^2c^2d^2L|,|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL\\ &-\frac{c^2(d-c)^2}{d^3}|,|ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)-abc^2L\\ &-a^2cd[ab^2+(d-c)]|,|L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL|\}\end{aligned}$$

Therefore,

$$\begin{aligned}V &\geq K_1(x^2+y^2+z^2+w^2)\\ &+\frac{\gamma}{2\tau}\int_{-\tau}^{0}[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)+w^2(t+\theta)]d\theta\end{aligned} \quad (3.7)$$

By using the the Schwartz inequality $|xy| \leq \frac{1}{2}\left|x^2+y^2\right|$, on the equation (3.2), we have

$$\begin{aligned}2V_1(x,y,z,w) \leq \left(\frac{\delta}{\Delta}\right)&\{[A+E+F+G]x^2+[B+E+H+I]y^2+[C+F+H+J]z^2\\ &+[D+G+I+J]w^2\} \leq K_2(x^2+y^2+z^2+w^2),\end{aligned} \quad (3.8)$$

where

$$K_2 = \left(\frac{\delta}{\Delta}\right)$$
$$\max\left\{[A+E+F+G],[B+E+H+I],[C+F+H+J],[D+G+I+J]\right\}.$$

Therefore,

$$V \le K_2(x^2+y^2+z^2+w^2) + \frac{\gamma}{2\tau}\int_{-\tau}^{0}[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)+w^2(t+\theta)]d\theta. \quad (3.9)$$

The R.H.S. of the inequalities (3.6) and (3.9) are always positive, hence by the definition of K_1 and K_2, V is positive definite and so we have

$$K_1(x^2+y^2+z^2+w^2) \le V(t;x,y,z) \le K_2(x^2+y^2+z^2+w^2), \quad (3.10)$$

which proves Lemma 3.1. □

Lemma 3.2. *In addition to assumptions of Theorem 2.1, let the condition (v) of the Theorem 2.2 be satisfied also. Then there are positive constants* $K_j = K_j(\alpha,\beta,\rho,\Delta,\gamma,\delta,\tau)(j=3,4)$ *such that for any solution* (x,y,z) *of the system* (1.3),

$$\dot V|_{(1.3)} \equiv \frac{d}{dt}V|_{(1.3)}(t;x,y,z,w) \le -K_3(x^2+y^2+z^2+w^2) + K_4\left(x^2+y^2+z^2+w^2\right)^{\frac{1}{2}} M(t). \quad (3.11)$$

Proof. By the definition of V we have that $\dot V = \dot V_1 + \dot V_2$.
From equations (1.1) and (1.3) we have,

$$\dot V_{1.3} = \frac{\partial V}{\partial x}\dot x + \frac{\partial V}{\partial y}\dot y + \frac{\partial V}{\partial z}\dot z + \frac{\partial V}{\partial w}\dot w = \frac{\partial V}{\partial x}y + \frac{\partial V}{\partial y}z + \frac{\partial V}{\partial z}w + \frac{\partial V}{\partial z}\left(-aw - f(x,y)z - g(y) - h(x) + M(t)\right)$$

after some simplification,we have that

$$\dot V_1 = \left(\frac{\delta}{\Delta}\right)\left\{-Gh(x)x - Ig(y)y - [Jb-H]z^2 - [Da-J]w^2 - Gg(y)x - Ih(x)y - [Gb-E]xz - Jh(x)z - [Ga-F]xw - Dh(x)w - [Ib-F-B]yz - Jg(y)z - [Ia-G-H]yw - Dg(y)w - [Db+Ja-I-C]zw + Ey^2 + Axy + p(t)[Gx+Iy+Jz+Dw]\right\}. \quad (3.12)$$

on using the hypotheses on f, g and h, we have

$$\dot{V}_1 \le \left(\frac{\delta}{\Delta}\right)\big\{-Gdx^2 - [Ic-E]y^2 - [Jb-H]z^2 - [Da-J]w^2 - [Gc+Id-A]xy - [Gb+Jd-E]xz - [Ga+Dd-F]xw - [Ib+Jc-F-B]yz - [Ia+Dc-G-H]yw - [Db+Ja-I-C]zw + [h(0)+g(0)+M(t)][Gx+Iy+Jz+Dw]\big\} \quad (3.13)$$

and this is equivalent to

$$\dot{V}_1 \le \left(\frac{\delta}{\Delta}\right) \times \big\{-K_3(x^2+y^2+z^2+w^2) + [h(0)+g(0)+M(t)][Gx+Iy+Jz+Dw]\big\}$$

where $K_3 = \max\{Gd, [Ic-E], [Jb-H], [Da-J]\}$.

Inequality (3.13) further reduces to

$$\dot{V}_1 \le \left(\tfrac{\delta}{\Delta}\right)\big\{-K_3(x^2+y^2+z^2+w^2) + K_4(|x|+|y|+|z|+|w|)M(t)\big\} \quad (3.14)$$

with $K_4 = \max\{D, G, I, J\}$.

Also from V_2 we have that

$$\begin{aligned}\dot{V}_2 \le & \frac{\gamma}{2\tau}\int_{-\tau}^{0}[x^2(t) - x^2(t+\theta) + y^2(t) - y^2(t+\theta) + z^2(t) - z^2(t+\theta) \\ & + w^2(t) - w^2(t+\theta)]d\theta \\ = & \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t) + y^2(t) + z^2(t)) + w^2(t)d\theta \\ & - \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)d\theta.\end{aligned}$$

Therefore,

$$\dot{V}_2 \le \gamma(x^2(t) + y^2(t) + z^2(t) + w^2(t)) - \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)d\theta. \quad (3.15)$$

Combining inequalities (3.14) and (3.15) for $\gamma > 0$, we have that

$$\dot{V} \le -(K_3 - \gamma)(x^2 + y^2 + z^2 + w^2)$$

$$- \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)d\theta$$
$$+ K_4(|x|+|y|+|z|+|w|)M(t). \quad (3.16)$$

for $K_3 > \gamma$ we have

$$\dot{V} \leq -K_5(x^2+y^2+z^2+w^2)$$
$$- \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)d\theta$$
$$+ K_4(|x|+|y|+|z|+|w|)M(t). \quad (3.17)$$

where

$$K_5 = K_3 - \gamma > 0$$

$$\dot{V} \leq -K_5(x^2+y^2+z^2) + K_4(|x|+|y|+|z|+|w|)M(t). \quad (3.18)$$

But

$$(|x|+|y|+|z|+|w|) \leq 2\left(x^2+y^2+z^2+w^2\right)^{\frac{1}{2}}.$$

This makes the inequality (3.18) to become

$$\dot{V} \leq -K_5(x^2+y^2+z^2+w^2) + K_6\left(x^2+y^2+z^2+w^2\right)^{\frac{1}{2}}M(t). \quad (3.19)$$

where

$$K_6 = 2K_4$$

This completes the proof of Lemma 3.2. □

4. Proof of Main Results

We shall now give the proofs of the main results.

Proof of Theorem 2.1. The proof of Theorem 2.1 follows from Lemmas 3.1 and 3.2 where it has been established that the trivial solution of the equation (1.1) is uniformly asymptotically stable. i.e every solution $(x(t), \dot{x}(t), \ddot{x}(t), \dddot{x})$ of the system (1.3) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2 \longrightarrow 0$ as $t \longrightarrow \infty$. □

Proof of Theorem 2.2. Clearly from the inequalities (3.7) and (3.9), we have that

$$V \geq K_1(x^2+y^2+z^2+w^2) \quad (4.1)$$

Combining inequalities (4.1) and (3.9) we have

$$K_1(r^2) \le V(t,x,y,z,w,x_t,y_t,z_t,w_t) \le K_2(r^2) + \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta)d\theta, \quad (4.2)$$

where $r^2 = x^2+y^2+z^2+w^2$.

Indeed from the inequality (3.19),

$$\frac{dV}{dt} \le -K_5V + K_6(x^2+y^2+z^2+w^2)^{\frac{1}{2}}\left|M(t)\right|.$$

Also from equation (3.10),

$$K_1(x^2+y^2+z^2+w^2) \le V$$

$$\Rightarrow (x^2+y^2+z^2+w^2)^{\frac{1}{2}} \le \left(\frac{V}{K_1}\right)^{\frac{1}{2}}$$

Thus inequality (3.19) becomes

$$\frac{dV}{dt} \le -K_5V + K_7V^{\frac{1}{2}}\left|M(t)\right| \quad (4.3)$$

where

$$K_7 = \frac{K_6}{\sqrt{K_1}}$$

This can be written as

$$\dot{V} \le -2K_8V + K_7V^{\frac{1}{2}}\left|M(t)\right|. \quad (4.4)$$

with

$$K_8 = \frac{1}{2}K_5$$

Therefore

$$\dot{V} + K_8V \le -K_8V + K_7V^{\frac{1}{2}}\left|M(t)\right|. \quad (4.5)$$

$$\le K_7V^{\frac{1}{2}}\left\{\left|M(t)\right| - K_9V^{\frac{1}{2}}\right\}, \quad (4.6)$$

where

$$K_9 = \frac{K_8}{K_7}$$

Thus inequality (4.6) becomes,

$$\dot{V} + K_8 V \leq K_7 V^{\frac{1}{2}} V^*, \tag{4.7}$$

where

$$V^* = \left\{ |M(t)| - K_9 V^{\frac{1}{2}} \right\} \tag{4.8}$$
$$\leq |M(t)|$$

when $|M(t)| \leq -K_9 V^{\frac{1}{2}}$

$$V^* \leq 0, \tag{4.9}$$

and when $|M(t)| \geq K_9 V^{\frac{1}{2}}$,

$$V^* \leq |M(t)| \cdot \frac{1}{K_9} \tag{4.10}$$

Substituting (4.10) into (4.7) we have

$$\dot{V} + K_8 V \leq K_{10} V^{\frac{1}{2}} |M(t)|, \tag{4.11}$$

where $K_{10} = \dfrac{K_7}{K_9}$.

This implies that (4.11) can be put as

$$V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \leq K_{10} |M(t)|, \tag{4.12}$$

Multiplying both sides of the inequality (4.12) by $e^{\frac{1}{2} K_8 t}$, gives

$$e^{\frac{1}{2} K_8 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \right\} \leq e^{\frac{1}{2} K_8 t} K_{10} \left| M(t) \right|, \tag{4.13}$$

i.e

$$2 \frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2} K_8 t} \right\} \leq e^{\frac{1}{2} K_8 t} K_{10} \left| M(t) \right|. \tag{4.14}$$

Integrating both sides of (4.14) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2} K_8 \omega} \right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2} K_8 \tau} K_{10} \left| M(\tau) d\tau \right|, \tag{4.15}$$

which implies that

$$\left\{ V^{\frac{1}{2}}(t) \right\} e^{\frac{1}{2} K_8 t} \leq V^{\frac{1}{2}}(t_0) e^{\frac{1}{2} K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} \left| M(\tau) \right| e^{\frac{1}{2} K_8 \tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2} K_8 t} \left\{ V^{\frac{1}{2}}(t_0) e^{\frac{1}{2} K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} \left| M(\tau) \right| e^{\frac{1}{2} K_8 \tau} d\tau \right\}.$$

On utilizing inequalities (3.6) and (3.8), we have

$$\begin{aligned} &K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t)) \\ &\leq e^{-K_8 t}\Big\{(K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)++\dddot{x}^2(t_0)))^{\frac{1}{2}}e^{\frac{1}{2}K_8 t_0} \\ &\qquad + \frac{1}{2}K_{10}\int_{t0}^{t}|M(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\Big\}^2, \end{aligned} \tag{4.16}$$

for all $t \geq t_0$.

Thus,

$$\begin{aligned} x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) &\leq \frac{1}{K_1}\Big\{e^{-K_8 t}\Big\{(K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0) \\ &\quad +\dddot{x}^2(t_0)))^{\frac{1}{2}}e^{\frac{1}{2}K_8 t_0}+\frac{1}{2}K_{10}\int_{t0}^{t}|M(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\Big\}^2\Big\} \\ &\leq \Big\{e^{-K_8 t}\Big\{A_1+A_2\int_{t0}^{t}|M(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\Big\}^2\Big\}, \end{aligned} \tag{4.17}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, \ldots K_{10}$ and $(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}^2(t_0))$.

By substituting $K_8 = \sigma$ in the inequality (4.20), we have

$$\begin{aligned} &x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}^2(t) \\ &\qquad \leq \left\{e^{-\sigma t}\left\{A_1+A_2\int_{t0}^{t}|M(\tau)|\,e^{\frac{1}{2}\sigma\tau}d\tau\right\}^2\right\} \leq K \end{aligned} \tag{4.18}$$

for sufficiently large t where K is a constant This completes the proof. □

Proof of Theorem 2.3. From the definition of function V and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can be obtained as

$$V \geq K_1\left(x^2+y^2+z^2+w^2\right), \tag{4.19}$$

and since $P \neq 0$ we can revise the conclusion of Lemma 3.2, i.e,

$$\dot{V} \leq -K_3(x^2+y^2+z^2+w^2)+K_5(|x|+|y|+|z|+|w|)\,|M|\,,$$

and we obtain by using the condition (vi) as stated in the Theorem 2.3 that

$$\dot{V} \leq K_5(|x|+|y|+|z|+|w|)^2\theta(t). \tag{4.20}$$

Using the inequality $|xy|\leq \frac{1}{2}(x^2+y^2)$ on (4.20), we have

$$\dot{V} \leq K_{11}(x^2+y^2+z^2+w^2)\theta(t), \tag{4.21}$$

where $K_{11} = 2K_5$.

From inequalities (4.19) and (4.21) we have,

$$\dot{V} \leq K_{11}V\theta(t). \tag{4.22}$$

Integrating inequality (4.22) from 0 to t gives

$$V(t) - V(0) \leq K_{12}\int_0^t V(s)\theta(s)ds, \tag{4.23}$$

where $K_{12} = \frac{K_{11}}{K_1} = \frac{2K_5}{K_1}$. Thus,

$$V(t) \leq V(0) + K_{12}\int_0^t V(s)\theta(s)ds. \tag{4.24}$$

Applying the Grownwall-Reid-Bellman theorem on the inequality (4.24) yields

$$V(t) \leq V(0)exp\left(K_{12}\int_0^t \theta(s)ds\right). \tag{4.25}$$

This completes the proof of Theorem 2.3. □

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