International Journal of Differential Equations and Applications
Volume 11 No. 4 2012, 227-239
ISSN: 1311-2872 (printed version)
url: http://www.ijpam.eu

acadpubl.eu

CHARGED PARTICLES THROUGH ELECTROMAGNETIC FIELDS

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Abstract: In this paper we consider the mathematical model of charged particles passing through electromagnetic fields. A system of nonlinear ODEs describes the physical phenomena. Some known results for existence, uniqueness and stability are used to describe behavior of the solutions. The main statement contains some estimates for the forces acting on the moving particles as for this purpose the existence and uniqueness statements are used.

AMS Subject Classification: 34A12, 83C50, 34A34, 97M50, 83A05
Key Words: Maxwell equations, estimates, Cauchy problem, ordinary differential equations, electromagnetic fields

1. Introduction

In this paper we consider a mathematical model of charged particles with relativistic mass passing through electromagnetic fields denoted by E (electric field) and B (magnetic field). There is a number of publications devoted to this topic.

Received: October 3, 2012

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For all classical results concerning this topic we refer the reader to [3]-[5]. We show here the dynamics of moving particles, and provide explicit examples of electric field, which is orthogonal to the particle speed or when it is zero.

The mathematical model considered here describes a beam of charged particles moving in some domain. Such a model can be seen for instance in [6] and [7] (see, e.g. [3], [4], [5]). Our motivation for this paper comes from the question how the existence and uniqueness statements can be applied for estimation of the dynamics of the considered physical phenomena. Consider some physical quantities. The speed of the particle on its trajectory is denoted by $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$, and the electric and magnetic fields by $E = (E_1, E_2, E_3) \in \mathbb{R}^3$, $E_i = E_i(t, x_1, x_2, x_3)$ $(i = 1, 2, 3)$ and $B = (B_1, B_2, B_3) \in \mathbb{R}^3$, $B_i = B_i(t, x_1, x_2, x_3)$ $(i = 1, 2, 3)$, respectively.

In Section 2 we discuss some properties of the model describing the moving particles.

Some statements for existence, uniqueness, stability and instability (in the sense of Lyapunov) for the model under consideration which stem from the classical statements of Picard-Lindelöf and Kamke are considered in Section 3.

In Section 4 some estimates to the forces acting on the moving particles are shown, as for this purpose we use the conditions to the statements of Picard-Lindelöf and Kamke.

2. The Mathematical Model

Introduce some standard quantities characterising the electric and magnetic phenomena. Consider $E = E(t, x, y, z)$ and $B = B(t, x, y, z)$, electric and magnetic fields, respectively, which are some vector functions depending on the time and space $(t, \mathbf{r})$ $(\mathbf{r} = (t, x, y, z))$. Denote the charge of the particle or field by $q = q(t, \mathbf{r})$ (scalar), the charge density by $\rho = \rho(t, \mathbf{r})$ (scalar), and current density by J (vector). Then the Maxwell's equations have the classical form,

$$\begin{aligned} \nabla \cdot E &= \frac{\rho}{\varepsilon} \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times B &= \mu J + \mu\varepsilon \frac{\partial E}{\partial t}, \end{aligned} \tag{1}$$

where ε is the permittivity, and μ is the permeability. Making use of Maxwell equations (1) and after adding some ODEs describing the movement and interaction of the charged particles we construct a mathematical model of the passing through the electromagnetic fields particles. The estimation of the forces in this model under some requirements to the known electromagnetic fields is our main goal in this paper.

In order to establish the mathematical model of the relativistic dynamics we take advantage of Einstein theory of relativity, known as Special Relativity (SR). The energy $\mathcal{E}$ and momentum $p = (p^k)$ $(k = 1, 2, 3)$ of a particle with rest mass m_0, moving with velocity $v = (v^k)$ $(k = 1, 2, 3)$ w.r.t. a given frame of reference in accordance with SR have the form

$$\mathcal{E} = \gamma m_0 c^2, \qquad p = \gamma m_0 v \tag{2}$$

respectively, where γ is the Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$, c is the light speed that is a known constant. The quantity $m \equiv \gamma m_0$ is the relativistic mass of the object in the given frame of reference. The energy and momentum of a particle with invariant mass m_0 are related by the formulas

$$\mathcal{E}^2 - (pc)^2 = (m_0 c^2)^2, \qquad pc^2 = \mathcal{E} v^2 \tag{3}$$

The first equality in (3) is referred to as the relativistic energy-momentum equation, while the energy $\mathcal{E}$ and the momentum p depend on the frame of reference in which they are measured. The quantity $\mathcal{E}^2 - (pc)^2$ is invariant, being equal to the squared invariant mass of the object (up to the multiplicative constant c^4) (see, e.g. [5]). Thus following the analogy with the Newton second law in classical mechanics $F = m_0 w$ (w is the acceleration of the particle), we reach to the formula $F = \dfrac{dp}{dt}$, where the force $F = (F^\alpha)$ $(\alpha = 0, 1, 2, 3)$, and $p = \gamma m_0 v$ is the momentum. Thus, the force F has the form

$$F = m_0 \frac{d(\gamma v)}{dt} = m_0 \left(\frac{d\gamma}{dt} v + \frac{dv}{dt} \gamma \right),$$

and

$$F = \frac{\gamma^3 m_0 (v \cdot w)}{c^2} v + \gamma m_0 w. \tag{4}$$

Here we use the mathematical model introduced in [6]. Consider the movement of a single charged particle with a static mass m_0, i.e. the mass of the particle in the moving coordinate system. Then the relativistic mass $m = m_0/\sqrt{1 - y^2/c^2}$, where assume that $y = (y_1, y_2, y_3)$ is the three dimensional vector of the velocity to the particle, and c is the habitual light velocity. Denote further by $x = (x_1, x_2, x_3)$ the position vector of the same particle. Then the dynamics of this physical system could be expressed by the following system of ODEs:

$$\begin{aligned} &\dot{x} = y, \\ &(\dot{my}) = E + y \times B + G(t, x, y) \end{aligned} \tag{5}$$

with an initial condition at the moment $t_0 \geq 0$,

$$x(t_0) = x_0, \quad y(t_0) = y_0. \tag{6}$$

Here $G = G(t, x, y)$ is a vector-function that does not depend explicitly on the components of the electromagnetic field. It expresses the influence of some potential and dissipative forces stemming from the interaction of the moving particles with the environment. Also the gravitational effects and radiation friction are taken into consideration. Let $g = g(t, x)$ be some differentiable function of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then define the total derivative of g through any continuous vector field $f(t, x)$ in the sense that for given system of ODEs

$$\frac{dx}{dt} = f(t, x), \quad x = (x_1, \ldots, x_n), \quad f = (f_1, \ldots, f_n)$$

we differentiate g through the trajectory of the above system. Thus obtain

$$\dot{g} \equiv \frac{\partial g}{\partial t} + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} f_i.$$

Furthermore, assume that $n = 3$. Denote by $y = (y_1, y_2, y_3)$ the three dimensional speed vector to the considered particle at the point $x = (x_1, x_2, x_3)$ and require that E, B and G to be sufficiently smooth functions. Note that if the right hand side of the system (5) is Lipschitzean then there exists a unique solution determining a unique movement of the considered particle. Here the following question would be of interest: If there exists some electromagnetic field (E and B) propagating a corresponding particle's movement that can be determined by the system (5) with a certain and preliminary known velocity distribution $\eta = \eta(t, x)$. In order to answer to this question we define the system,

$$\dot{x} = \eta(t, x) \tag{7}$$

at the same initial data $x(t_0) = x_0$ as it was in (6). Here η is the known vector field.

Denote by $Lip(\Omega)$, $\Omega \subset \mathbb{R}^3$ the class of all Lipschitzean vector-functions $f(x)$, defined on some domain $\Omega \subset \mathbb{R}^3$.

Suppose that the following hypotheses hold:

H1. The vector-function $\eta = \eta(t, x) \in \mathbb{R}^3$ is Lipschitzean w. r. t. x, i.e. $\eta(t, \cdot) \in Lip(\Omega)$

$$|\eta(t, \tilde{x}) - \eta(t, \hat{x})| \le K|\tilde{x} - \hat{x}| \ \ \forall t \ge t_0, \tag{8}$$

for any pair of vectors $\tilde{x}$ and $\hat{x}$ in Ω with real components. By $|\cdot|$ denote the common Euclidean norm in $\mathbb{R}^3$.

H2. The quantities E, B possess continuous partial derivatives of first order, and G is continuous vector-function w. r. t. all variables t, x and y.

We shall take advantage of the following statements being seen in [6] (Chapter 2):

Theorem 2.1. (see [6]) *Let H1 be satisfied to the vector field (7). Then there exist vector-functions E and B satisfying H2 and Maxwell equations (1) such that the system (5) determines the movement of the particle through the electromagnetic field, corresponding to the movement determined by the system (7) under the same initial data (6).*

Next we take into account Gauss' law for magnetism in the Maxwell' system (1)

$$\text{div } B \equiv \nabla \cdot B = 0,$$

and following the classical theory of electromagnetism we include the vector-potential $\mathbf{A} = \mathbf{A}(t, x)$ in our considerations, which is differentiable vector-function w. r. t. its arguments. Thus one may let

$$B = \nabla \times \mathbf{A} \equiv \text{rot } \mathbf{A},$$

and then from the first equation in (1) (Gauss' law) obtain $\nabla \times \left[E + \dfrac{\partial \mathbf{A}}{\partial t}\right] = 0$. Introducing the scalar function $\varphi = \varphi(t, x)$, known as electric potential, obtain the equation $E + \dfrac{\partial \mathbf{A}}{\partial t} = -\nabla\varphi$.

Note. Using SR notation we may insert in (1) the four-vector potential $(\varphi, \mathbf{A})$.

By the above stated potentials the system (5) takes the form

$$\begin{aligned} &\dot{x} = y, \\ &(\dot{my}) = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} + y \times \text{rot } \mathbf{A} + G(t, x, y) \end{aligned} \tag{9}$$

under the initial data in (6). The following statement shows the correspondence between four-vector potential $(\varphi, \mathbf{A})$ and the moving particle described by the system (7) under the initial data $x(t_0) = x_0$ given in (6).

Theorem 2.2. ([6]) *Assume that the conditions in Theorem 2.1 are satisfied. Then there exists four-vector potential $(\varphi, \mathbf{A})$ propagating an electromagnetic field by which the charged particle moves and satisfies the system (9). This movement of the particle corresponds to the movement determined by the system (7) under the initial data $x(t_0) = x_0$ stated in (6).*

Following the theory presented in [6] and [7] the question about uniqueness of the solution of (7) under initial condition $x(t_0) = x_0$ in (6) would be of interest.

H3. Assume that the vector-function $\eta(t, x)$ is continuous in the parallelepiped

$$\Pi = \{(t, x): \ t_0 \leq t \leq t_0 + a, \ |x - x_0| \leq b\},$$

Lipschitzean in x, and define the number α as

$$\alpha \ = \ \min\{a, b/M)\},$$

where M is an upper limit for $|\eta(t, x)|$ on Π.

Theorem 2.3. *Let H3 be satisfied. Then there exists a unique solution $x = x(t)$, $t \in [t_0, t_0 + \alpha]$ for the system (7) under initial condition $x(t_0) = x_0$ in (6).*

The proof follows directly from the Picard-Lindelöf's theorem (see, e.g. [2], Chapter 2).

Remark. If in H3 we require for $\eta(t, x)$ to be only continuous on Π, i.e. the Lipschitz condition for $\eta(t, x)$ in x to be removed, then the solution of (7) under the initial condition $x(t_0) = x_0$ stated in (6) solely exists. However, we could not say whether it is unique. Therefore the above theorem states only existence of the solution, according to the Peano theorem (see, e.g. [2], Chapter 2).

Another statement ensuring uniqueness is the known Kamke uniqueness theorem that is a most general (see, e.g. [2]). To formulate Kamke statement we make use of the set Π defined in H3 and then state:

Theorem 2.4. ([2]) *Let $\eta(t, x)$ is continuous in Π, and $\omega(t, u)$ be continuous scalar function in $\Pi_0 : \ t_0 < t \leq t_0 + a$, $0 \leq u \leq 2b$ having the properties: 1) $\omega(t, 0) = 0$; 2) the unique solution $u = u(t)$ of the ODE*

$$u' = \omega(t, u),$$

satisfying the conditions

$$u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{t - t_0} \to 0 \quad \text{as} \quad t \to t_0 + 0,$$

$$\text{on} \quad (t_0, t_0 + \varepsilon] \quad \text{and for any} \quad \varepsilon > 0$$

is solely the function $u(t) \equiv 0$*. If for pair of points* $(t, \tilde{x}), (t, \hat{x}) \in \Pi$*,* $t > t_0$*,*

$$|\eta(t, \tilde{x}) - \eta(t, \hat{x})| \leq \omega(t, |\tilde{x} - \hat{x}|),$$

then the Cauchy problem (7), (6) on any interval $[t_0, t_0 + \varepsilon]$ *possesses a unique solution.*

Remark. In the above stated theorem by $|z|$ we denote the norm (for instance, the Euclidean norm).

The real and continuous scalar function $V(t, x)$, defined in the domain $\|x\| \leq h$ $(h > 0)$, $t \geq 0$ which is a subset of $\mathbb{R}^1_+ \times \mathbb{R}^3$ and such that $V(t, 0) = 0$ $\forall t \geq 0$, is said to be positive definite if there exists a real continuous scalar function $W(x)$ defined for $\|x\| \leq h$ and such that: 1) $W(0) = 0$; 2) $W(x) > 0$ for $\|x\| \neq 0$; 3) $V(t, x) \geq W(x)$.

If the properties 2) and 3) to the function $W(x)$ are replaced by 2*) $W(x) < 0$ for $\|x\| \neq 0$ and 3*) $V(t, x) \leq W(x)$, respectively, then the function $V(t, x)$ is negative definite.

3. Stability and Instability

The following results give us some information about stability to the physical process of moving charged particles through the electromagnetic field (see, e.g. [7], 5, Theorem 11).

Theorem 3.1. (stability) *The zero solution of the system (7) is stable in Liapunov sense iff there exists a real scalar function* $V(t, x)$ *defined in the domain* $\|x\| \leq h_0$ $(t \geq 0,\ h_0 > 0)$ *and satisfying following conditions: 1) Continuous in* x *at the point* $x = 0$*, and* $V(t, 0) = 0$*,* $\forall t \geq 0$*; 2)* $V(t, x)$ *is positive definite; 3) The function* $V_1(t, t_0, x_0) = V(t, x(t, t_0, x_0))$ *is nonincreasing for* $t \geq t_0 \geq 0$*,* $\forall x_0$ *such that* $\|x_0\| \leq h_0$*.*

Due to the Consequence 1 and 2 in [7] (Paragraph 5) it turns out that

$$\frac{dV}{dt} = W \leq 0$$

everywhere. Therefore, the condition 3) in Theorem 3.1 could be verified directly by computing total derivative of V through the integral curves. From Theorem 3.1 follows that $dV/dt \leq 0$ hence the zero solution of (7) should be stable uniformly w.r.t. $t_0 \geq 0$, i.e. the definition of stability (in the sense of Liapunov) does not depend on the point t_0.

The following statement concerns the asymptotic stability (see, e.g. [7]).

Theorem 3.2. (asymptotic stability) *Let H1 be satisfied and let $V(t,x)$ be the positive definite function, continuous in x at the point $x = 0$ uniformly w.r.t. $t \geq 0$ and such that*

$$\frac{dV}{dt} = -W(t,x),$$

where the function $W(t,x)$ is positive definite function. Then the equilibrium point $x = 0$ is asymptotic stable (in the sense of Liapunov) uniformly in $t_0 \geq 0$.

The following theorem gives us a necessary and sufficient condition for the instability uniformly in $t_0 \geq 0$ (see, e.g. [7]).

Theorem 3.3. (instability) *Let H1 be satisfied. Then the zero solution of (7) be unstable iff there exist a pair of scalar function $V(t,x)$ and $W(t,x)$ in the domain $\|x\| \leq h_0$, $t \geq 0$ such that: 1) $V(t,x)$ is bounded in some vicinity of the point $x = 0$, $\forall t \geq 0$, i.e., $V(t,x) < l$ for $\|x\| \leq h_0$, $t \geq 0$; 2) There exists at least one point $\overline{x}$ in a small vicinity U_0 of $x = 0$ and such that $V(t_0, \overline{x}) > 0$ for some $t_0 \geq 0$; 3) The total derivative of the function $V(t,x)$ (concerning the system (7)) satisfies the equality*

$$\frac{dV}{dt} = \lambda V + W(t,x),$$

where $\lambda > 0$, and $W(t,x)$ is a real, continuous and nonnegative function for $\|x\| \leq h_0$.

A case of nonrelativistic particle.
Consider the case when the particle speed $|\eta|$ is comparatively small w. r. t. the light speed c (see, e.g. [3]). Then the dynamics of the particle may have the form $m_0 \dfrac{d\eta}{dt} = qE + \dfrac{q}{c}\eta \times B$. Consider the static homogeneous magnetic field B, and $E = 0$. Assume that the axis Oz is parallel to B. Thus obtain

$$d\eta_1/dt = \omega_L \eta_2, \quad d\eta_2/dt = -\omega_L \eta_1, \quad d\eta_3/dt = 0,$$

where $\omega_L = qB/m_0 c$ is cyclotron (Larmor) frequency. Here we choose some initial speed v_0. By the method of complex analysis we get the solution

$$\eta_1 = \eta_{01} \cos \omega_L t + \eta_{02} \sin \omega_L t$$

$$\eta_2 = \eta_{02} \cos \omega_L t - \eta_{01} \sin \omega_L t.$$

Note. We emphasize that very often some authors take for q and c dimensionless quantity, $q = 1$ and $c = 1$, thus everywhere one may get dimensionless results.

Consider the system (7). In most of cases there is a vicinity of the zero solution such that

$$\eta(t, x) = Q(t)x + \mathcal{O}(t, x), \tag{10}$$

where $\mathcal{O}(t, x) \to 0$, $\forall t \geq 0$ as $\|x\| \to 0$, $Q(t)$ is a linear operator (3×3 matrix) with entries $q_{ij}(t)$ $(i, j = 1, 2, 3)$ which are real continuous and bounded functions $\forall t \geq 0$. Thus the system

$$\frac{d\tilde{x}}{dt} = Q(t)\tilde{x} \tag{11}$$

is a linear approximation of (10), i.e. if the zero solution of (11) is asymptotically stable, then the zero solution of (10) is also asymptotically stable.

Announce the following condition:

A. The function $x(t)$ satisfies the decay rate condition

$$a_1 \|x_0\| e^{-b_1(t-t_0)} \leq \|x(t)\| \leq a_2 \|x_0\| e^{-b_2(t-t_0)} \quad \text{for} \quad t > t_0,$$

where $a_i, b_i > 0$ $(i = 1, 2)$.

Furthermore, we consider the following statement that the reader can find in [6]:

Theorem 3.4. *The solution $\tilde{x}(t)$ of (11) with initial condition $\tilde{x}(t_0) = x_0$ satisfies the condition A iff when there exist two quadratic forms $V(t, \tilde{x})$ and $W(t, \tilde{x})$ such that:*
1) $V(t, \tilde{x})$ is positive-definite function, $W(t, \tilde{x})$ is negative-definite, and

$$\alpha_1 \|\tilde{x}\|^2 \leq V(t, \tilde{x}) \leq \alpha_2 \|\tilde{x}\|^2$$

$$-\beta_1 \|\tilde{x}\|^2 \leq W(t, \tilde{x}) \leq -\beta_2 \|\tilde{x}\|^2,$$

where $a_i, b_i, \alpha_i, \beta_i > 0$ $(i = 1, 2)$.
2) The pair of functions V and W satisfy the differential equality $dV/dt = W$.

Assume that the quadratic forms $V(t,x)$ and $W(t,x)$ are defined by $V = x^*(t)\tilde{V}x(t)$ and $W = x^*(t)\tilde{W}x(t)$, respectively, hence the differential equality should take the form $dV/dt = (dx^*/dt)\tilde{V}x + x^*(d\tilde{V}/dt)x + x^*\tilde{V}dx/dt$. Having in mind the equation (11) thus obtain the equality $dV/dt = x^*\Big(Q^*(t)\tilde{V}(t) + d\tilde{V}/dt + \tilde{V}Q(t)\Big)x = x^*\tilde{W}x$ which is true on each trajectory of the system, i.e.

$$\frac{d\tilde{V}}{dt} + Q^*\tilde{V} + \tilde{V}Q = \tilde{W}. \tag{12}$$

The latter is a matrix equation known as *Lyapunov equation.* In the case $Q \equiv \{\text{const matrix}\}$ it follows that also $\tilde{V} \equiv \{\text{const matrix}\}$, and then from (12) get an algebraic matrix equation,

$$Q^*\tilde{V} + \tilde{V}Q = \tilde{W}. \tag{13}$$

The equation (13) is known and applicable in many sciences, and there are methods to be solved. Thus the existence and uniqueness of the solution of (13) is guaranteed if the eigenvalues of Q satisfy the equality $\lambda_i + \lambda_j \neq 0 \ \forall \ i$ and j. The unique solution of (13) can be found in the form

$$\tilde{V} \ = \ -\int_0^\infty e^{Q^*t}\tilde{W}e^{Qt}dt$$

(see, e.g. [1]).

4. Relations between Forces and Velocities

We assume that the following hypothesis hold:
H4. Suppose that $\eta \in C^1(\Pi)$, and the following inequalities hold:

$$\frac{\partial(m\eta_j)}{\partial t}\,\frac{\partial(m\eta_k)}{\partial t} \geq 0 \quad \forall \ \ j \neq k \ \ (j,k = 1,2,3), \tag{14}$$

and

$$\left(\sum_{i=1}^{3}\eta_i\frac{\partial\eta_i}{\partial x^k}\right)\left(\sum_{i=1}^{3}\eta_i\frac{\partial\eta_i}{\partial x^j}\right) \geq 0 \quad \forall \ \ j \neq k \ \ (j,k = 1,2,3). \tag{15}$$

Proposition 4.1. *Let H3 and H4 be satisfied. Then for the force F, acting on the moving particle, the following estimate hold:*

$$|F| \leq C_1$$
$$\left\{\sum_{i=1}^{3}\left[\left|\frac{\partial\eta_i}{\partial t}\right| + \frac{1}{2}|\eta_i|\frac{\partial(|\eta|^2)}{\partial t}\right] + |\eta|\sum_{i=1}^{3}\left[|\eta_i|\sum_{j=1}^{3}\sum_{s=1}^{3}|\eta_j|\left|\frac{\partial\eta_j}{\partial x_s}\right| + \sum_{k=1}^{3}\left|\frac{\partial\eta_i}{\partial x_k}\right|\right]\right\}, \tag{16}$$

where $C_1 = \dfrac{m_0}{(1-\eta^2)^{3/2}}$, *and* $(t,x) \in \Pi$.

Proof. The proof follows from the estimates to the derivatives of the momentum $m\eta$, where η is the known vector field from (7). Hence obtain

$$|F| \equiv |(\dot{m\eta})| \leq \sqrt{\sum_{i=1}^{3}[\partial(m\eta_i)/\partial t]^2} + \sqrt{\sum_{i=1}^{3}\left(\sum_{k=1}^{3}[\partial(m\eta_i)/\partial x_k]\eta_k\right)^2}.$$

Further on, applying some elementary inequalities, and by the hypothesis H4, we get

$$|F| \leq \sum_{i=1}^{3}|\partial(m\eta_i)/\partial t| + |\eta|\sum_{j=1}^{3}|\nabla(m\eta_j)|^2 \leq$$

$$\leq C_1\left\{\sum_{i=1}^{3}\left[\left|\frac{\partial\eta_i}{\partial t}\right| + \frac{1}{2}|\eta_i|\frac{\partial(|\eta|^2)}{\partial t}\right] + |\eta|\sum_{i=1}^{3}\left[|\eta_i|\sum_{j=1}^{3}\sum_{s=1}^{3}|\eta_j|\left|\frac{\partial\eta_j}{\partial x_s}\right| + \sum_{k=1}^{3}\left|\frac{\partial\eta_i}{\partial x_k}\right|\right]\right\},$$

where from the statement follows.

Another estimate concerns the projection of the vector $G(t,x,y)$ on the direction defined by $\eta^0 = \eta/|\eta|$ - the unit vector through η, which satisfies (5).

Proposition 4.2. *Let the condition of Proposition 4.1 be satisfied, and the vector field E be directed orthogonal to η. Then for the projection G_{η^0} of G on the direction η^0 we have the estimate:*

$$|G_{\eta^0}| \leq C_1$$
$$\left\{\sum_{i=1}^{3}\left[c^2\left|\frac{\partial\eta_i}{\partial t}\right| + \frac{1}{2}|\eta_i|\frac{\partial(|\eta|^2)}{\partial t}\right] + |\eta|\sum_{i=1}^{3}\left[|\eta_i|\sum_{j=1}^{3}\sum_{s=1}^{3}|\eta_j|\left|\frac{\partial\eta_j}{\partial x_s}\right| + \sum_{k=1}^{3}\left|\frac{\partial\eta_i}{\partial x_k}\right|\right]\right\}.$$

Proof. The proof follows from the inequality

$$|\eta^0 \cdot G| = |G_{\eta^0}| \leq |F|,$$

and the result (16) in Proposition 4.1, whereby the statement follows.

Note. The last Proposition 4.2 holds true also in the case when the vector field $E = 0$.

Furthermore, we suppose that the point $x = 0$ is an equilibrium point, i.e. $\eta(t, 0) = 0$, $\forall t \geq 0$, and η satisfies the condition H3. Then making use of the Proposition 4.1 we state:

Theorem 4.3. *Let be supposed that $\eta(t, x)$ satisfies the conditions in Kamke's theorem (Theorem 2.4) and $\omega(t, u) < 1$ in Π. Then*

$$|F| < \frac{m_0}{1 - \omega^2(|x|)} \left\{ \|\frac{\partial \eta}{\partial t}\| + \|\eta\|^2 \frac{\partial}{\partial t}\|\eta\| + \|\eta\|(\|\eta\| + 1)\|\frac{\partial \eta}{\partial x}\| \right\},$$

where $\|\eta\| = \sum_{j=1}^{3} |\eta_j|$, $\|\frac{\partial \eta}{\partial t}\| = \sum_{j=1}^{3} |\frac{\partial \eta_j}{\partial t}|$, *and* $\|\frac{\partial \eta_j}{\partial x}\| = \sum_{i=1}^{3}\sum_{j=1}^{3} |\frac{\partial \eta_i}{\partial x_j}|$

Thus it turns out that the forces on the particles can be estimated by using for this purpose the known existence and uniqueness results, and the upper bounds depend substantially on the field η.

Acknowledgments

This work was supported by Project NIS-2012, a grant No. 10988, UCTM - Sofia.

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