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OSCILLATION THEOREMS FOR SECOND ORDER NEUTRAL DELAY AND ADVANCED DIFFERENCE EQUATIONS

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Abstract: In this paper we obtain some oscillatory behavior of solutions of the second order neutral difference equations of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x_{n-\ell} + v_n x_{n-m}^{\alpha} = 0, \quad n \in N_0$$

and

$$\Delta(r_n\Delta(x_n + p_n x_{n+k})) + q_n x_{n+\ell} + v_n x_{n+m}^{\alpha} = 0, \quad n \in N_0.$$

Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we study the oscillatory behavior of solutions of the second order neutral delay difference equation of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-k})) + q_n x_{n-\ell} + v_n x_{n-m}^{\alpha} = 0, \quad n \in N_0 \tag{1.1}$$

and the advanced difference equation of the form

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$$\Delta(r_n\Delta(x_n + p_n x_{n+k})) + q_n x_{n+\ell} + v_n x^{\alpha}_{n+m} = 0, \quad n \in N_0 \tag{1.2}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $N_0 = \{n_0 + 0,\ n_0 + 1,\ n_0 + 2,\ \ldots\}$, subject to the following conditions:

(C_1) $\{r_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty$;

(C_2) $\{p_n\}$ is a nonnegtive real sequence with $0 \le p_n \le p < \infty$;

(C_3) $\{q_n\}$ and $\{v_n\}$ are positive real sequences;

(C_4) $k,\ \ell$ and m are positive integers;

(C_5) α is a ratio of odd positive integers.

Let $\theta = \max\{k, \ell, m\}$. By a solution of equation (1.1) ((1.2)) we mean a real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$ and satisfying equation (1.1)((1.2)) for all $n \ge n_0$. A solution $\{x_n\}$ is said to to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years there has been much research activity concerning the oscillation of second order neutral type difference equations, see for example [1-4, 8-12, 14-21], and the reference cited therein.

In [1, 3, 4, 8, 12], the authors studied the oscillatory behavior of solutions of equation (1.1) when $v_n \equiv 0$ and in [9, 11, 16, 17, 20], the authors studied the same problem for the equation (1.1) when $q_n \equiv 0$. Therefore in this paper we discusses the oscillatory behavior of equation (1.1) which unified the results obtained for linear and nonlinear cases. Further the results obtained for the equation (1.2) is seems to be new even for the linear or nonlinear cases.

In Section 2, we establish sufficient conditions for the oscillation of all solutions of equation (1.1) and (1.2) and in Section 3, we present some examples to illustrate the main results. The results established in this paper are discrete analogue of that in [5].

2. Main Results

In this section we obtain sufficient condition for the oscillation of all solution of equation (1.1) and (1.2). To prove our main results we need the following lemmas.

Lemma 2.1. *If $A \ge 0, B \ge 0$ and $0 < \alpha \le 1$, then*

$$A^{\alpha} + B^{\alpha} \ge (A + B)^{\alpha}. \tag{2.1}$$

Lemma 2.2. *If $A \geq 0, B \geq 0$ and $\alpha \geq 1$, then*

$$A^\alpha + B^\alpha \geq \frac{1}{2^{\alpha-1}}(A+B)^\alpha. \tag{2.2}$$

For the proof of Lemmas 2.1 and 2.2, see [7].

Lemma 2.3. *If $0 < \alpha < 1$, ℓ is a positive integer and $\{p_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} p_n = \infty$, then every solution of equation*

$$\Delta x_n + p_n x_{n-\ell}^\alpha = 0, \tag{2.3}$$

is oscillatory.

Lemma 2.4. *Let $\alpha > 1$. If there exists a $\lambda > \frac{1}{\ell}\log\alpha$ such that*

$$\liminf_{n\to\infty}[p_n exp(-e^{\lambda n})] > 0, \tag{2.4}$$

then every solution of equation (2.3) is oscillatory.

For the proof of Lemmas 2.3 and 2.4, see [13].

Lemma 2.5. *If $\alpha = 1$ and*

$$\liminf_{n\to\infty} \sum_{s=n-\ell}^{n-1} p_s > \left(\frac{\ell}{\ell+1}\right)^{\ell+1}, \tag{2.5}$$

then every solution of equation (2.3) is oscillatory.

Lemma 2.6. *If $\{p_n\}$ is a nonnegative sequence of real numbers and let ℓ be a positive integer. Suppose that (2.5) holds, then the difference inequality*

$$\Delta x_n - p_n x_{n+\ell}^\alpha \geq 0, \quad n \in N_0, \tag{2.6}$$

cannot have eventually positive solutions.

Lemma 2.7. *If $\{p_n\}$ is a nonnegative sequence of real numbers and let $\ell \in \{2, 3, \ldots\}$ be such that $\sum_{s=n-\ell+1}^{n-1} p_s > 0$ for all large n. Then the difference inequality (2.6) has an eventually positive solution if and only if the difference equation*

$$\Delta u_n + p_n x_{n+\ell}^\alpha = 0, \quad n \in N_0, \tag{2.7}$$

has an eventually positive solutions.

For the proof of Lemmas 2.5 - 2.7, see [6].

Lemma 2.8. *If $\{x_n\}$ is a positive solution of equation* (1.1), *then the corresponding function $z_n = x_n + p_n x_{n-k}$ satisfies*

$$z_n > 0, \quad r_n \Delta z_n > 0, \quad \Delta(r_n \Delta z_n) < 0 \tag{2.8}$$

eventually.

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then $z_n = x_n + p_n x_{n-k} > 0$ for all $n \geq n_1 \geq n_0$. From the equation (1.1), we have

$$\Delta(r_n \Delta z_n) = -q_n x_{n-\ell} - v_n x_{n-m}^{\alpha} < 0.$$

Consequently, $r_n \Delta z_n$ is nonincreasing and thus either $r_n \Delta z_n > 0$ or $r_n \Delta z_n \leq 0$. If $r_n \Delta z_n \leq 0$ then for $n \geq n_1$, we have

$$r_n \Delta z_n \leq r_{n_1} \Delta z_{n_1} < 0.$$

Dividing the last inequality by r_{n_1} and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$z_n < z_{n_1} + r_{n_1} \Delta z_{n_1} \sum_{s=n_1}^{n-1} \frac{1}{r_s} \to -\infty \text{ as } n \to \infty,$$

which is a contradiction for the positivity of z_n. This completes the proof. □

Before stating the next theorem, let us define

$$Q_n = \min\{q_n, q_{n-k}\}, \quad V_n = \min\{v_n, v_{n-k}\} \text{ for } n \in N_0, \tag{2.9}$$

and

$$Q_n^* = Q_n \sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s}, \tag{2.10}$$

also

$$V_n^* = \begin{cases} V_n \left(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s} \right)^{\alpha} & \text{if } 0 < \alpha \leq 1, \\ V_n 2^{1-\alpha} \left(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s} \right)^{\alpha} & \text{if } \alpha > 1. \end{cases} \tag{2.11}$$

Theorem 2.1. *If the first order neutral difference inequality*

$$\Delta w_n + \frac{1}{1+p^{\alpha}} Q_n^* w_{n-\ell+k} + \frac{1}{(1+p^{\alpha})^{\alpha}} V_n^* w_{n-m+k}^{\alpha} \leq 0, \tag{2.12}$$

where $\{Q_n^*\}$ and $\{V_n^*\}$ be defined in (2.10) *and* (2.11) *respectively, has no positive solution, then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-k} > 0$ for all $n \geq n_1 \geq n_0 + \theta$. Then $z_n > 0$ is satisfies (1.1) and the function

$$z_{n-\ell} = x_{n-\ell} + p_{n-\ell} x_{n-k-\ell} \leq x_{n-\ell} + p x_{n-k-\ell}, \tag{2.13}$$

and

$$z_{n-m} = x_{n-m} + p_{n-m} x_{n-k-m} \leq x_{n-m} + p x_{n-k-m}. \tag{2.14}$$

From the equation (1.1) we have

$$\Delta(r_n \Delta z_n) + q_n x_{n-\ell} + v_n x_{n-m}^{\alpha} = 0, \tag{2.15}$$

and

$$p^{\alpha} \Delta(r_{n-k} \Delta z_{n-k}) + p^{\alpha} q_{n-k} x_{n-k-\ell} + p^{\alpha} v_{n-k} x_{n-k-m}^{\alpha} = 0. \tag{2.16}$$

Combining (2.15) and (2.16), then using (2.9) we get

$$\begin{aligned} \Delta(r_n \Delta z_n + p^{\alpha} r_{n-k} \Delta z_{n-k}) + Q_n(x_{n-\ell} + p^{\alpha} x_{n-k-\ell}) \\ + V_n(x_{n-m}^{\alpha} + p^{\alpha} x_{n-k-m}^{\alpha}) \leq 0. \end{aligned} \tag{2.17}$$

Substituting (2.13) and (2.14) in (2.17), and also applying Lemmas 2.1 and 2.2, we have

$$\Delta(r_n \Delta z_n + p^{\alpha} r_{n-k} \Delta z_{n-k}) + Q_n z_{n-\ell} + V_n z_{n-m}^{\alpha} \leq 0, \tag{2.18}$$

and

$$\Delta(r_n \Delta z_n + p^{\alpha} r_{n-k} \Delta z_{n-k}) + Q_n z_{n-\ell} + 2^{1-\alpha} V_n z_{n-m}^{\alpha} \leq 0. \tag{2.19}$$

Since $y_n = r_n \Delta z_n > 0$ is decreasing, we have

$$z_n \geq y_n \sum_{s=n_1}^{n-1} \frac{1}{r_s}. \tag{2.20}$$

Substituting (2.20) in (2.18) and (2.19), we get

$$\Delta(y_n + p^{\alpha} y_{n-k}) + Q_n y_{n-\ell} \sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s} + V_n y_{n-m}^{\alpha} \Big(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s} \Big)^{\alpha} \leq 0,$$

and

$$\Delta(y_n + p^{\alpha} y_{n-k}) + Q_n y_{n-\ell} \sum_{s=n_1}^{n-\ell-1} \frac{1}{r_s} + 2^{1-\alpha} V_n y_{n-m}^{\alpha} \Big(\sum_{s=n_1}^{n-m-1} \frac{1}{r_s} \Big)^{\alpha} \leq 0.$$

By using (2.10) and (2.11), we have

$$\Delta(y_n + p^\alpha y_{n-k}) + Q_n^* y_{n-\ell} + V_n^* y_{n-m}^\alpha \leq 0. \tag{2.21}$$

Define

$$0 < w_n = y_n + p^\alpha y_{n-k} \leq (1 + p^\alpha) y_{n-k}. \tag{2.22}$$

Substituting (2.22) in (2.21), we get $\{w_n\}$ is a positive solution of the following inequality

$$\Delta w_n + \frac{1}{1+p^\alpha} Q_n^* w_{n-\ell+k} + \frac{1}{(1+p^\alpha)^\alpha} V_n^* w_{n-m+k}^\alpha \leq 0,$$

which is a contradiction and the proof is now complete. □

Corollary 2.1. *Assume $\ell > k$ and $\ell < m$. If $\alpha = 1$, and*

$$\liminf_{n\to\infty} \sum_{s=n-m+k}^{n-1} (Q_s^* + V_s^*) > (1+p)\Big(\frac{m-k}{m-k+1}\Big)^{m-k+1} \tag{2.23}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. From Theorem 2.1, the oscillation of (1.1) provides that (2.12) has no positive solution. Assume $\{w_n\}$ is a positive solution of (2.12). Then w_n is decreasing and if $\ell < m$, then

$$w_{n-\ell} \geq w_{n-m}.$$

Using the last inequality in (2.12), we get that $\{w_n\}$ is a positive solution of the difference inequality

$$\Delta w_n + \frac{1}{1+p}(Q_n^* + V_n^*) w_{n-m+k} \leq 0. \tag{2.24}$$

Inview of condition (2.23), Lemma 2.5 implies that the inequality (2.23) has no positive solution, which is a contradiction. This completes the proof. □

Corollary 2.2. *Assume $\ell > k$ and $\ell > m$. If $\alpha = 1$, and*

$$\liminf_{n\to\infty} \sum_{s=n-\ell+k}^{n-1} (Q_s^* + V_s^*) > (1+p)\Big(\frac{\ell-k}{\ell-k+1}\Big)^{\ell-k+1} \tag{2.25}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 2.1 and hence the details are omitted. □

Corollary 2.3. *Assume* $q_n \equiv 0,\quad m > k$ *and* $0 < \alpha < 1$ *in equation* (1.1). *If*

$$\sum_{s=n_0}^{\infty} V_s^* = \infty \tag{2.26}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.3 in Theorem 2.1 and hence the details are omitted. □

Corollary 2.4. *Assume* $q_n \equiv 0,\quad m > k$ *and* $\alpha > 1$ *in equation* (1.1). *If there exists a* $\lambda > 0$ *such that* $\lambda > \frac{1}{m-k} log\alpha$ *and*

$$\lim_{s\to\infty} \inf[V_s^* exp(-e^{\lambda s})] > 0, \tag{2.27}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.4 in Theorem 2.1 and hence the details are omitted. □

Lemma 2.9. *If* $\{x_n\}$ *is a positive solution of equation* (1.2), *then the corresponding function* $u_n = x_n + p_n u_{n+k}$ *satisfies*

$$u_n > 0,\quad r_n\Delta u_n > 0,\quad \Delta(r_n\Delta u_n) < 0 \tag{2.28}$$

eventually.

Proof. The proof is similar to that of Lemma 2.8 and hence the details are omitted. □

Before stating the next theorem, we define

$$R_n = \min\{q_n, q_{n+k}\},\quad S_n = \min\{v_n, v_{n+k}\} \text{ for } n \in N_0, \tag{2.29}$$

and

$$R_n^* = \frac{1}{r_n}\sum_{s=n}^{\infty} R_s, \tag{2.30}$$

also

$$S_n^* = \frac{1}{r_n}\sum_{s=n}^{\infty} T_s. \tag{2.31}$$

Theorem 2.2. *If the first order difference inequality*

$$\Delta u_n - \frac{1}{1+p^{\alpha}}\Big(R_n^* u_{n+\ell} + S_n^* u_{n+m}\Big) \geq 0, \tag{2.32}$$

where $\{R_n^*\}$ *and* $\{S_n^*\}$ *be defined in* (2.30) *and* (2.31) *respectively for* $n \geq n_0 \in N_0$, *has no positive solution, then every solution of equation* (1.2) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.2). Without loss of generality we may assume that $x_n > 0$ and $x_{n+k} > 0$ for all $n \geq n_1 \geq n_0 + \theta$. Then $u_n = x_n + p_n x_{n+k}$ satisfies (1.2) and the function

$$u_{n+\ell} = x_{n+\ell} + p_{n+\ell}x_{n+\ell+k} \leq x_{n+\ell} + p x_{n+\ell+k}, \tag{2.33}$$

and

$$u_{n+m} = x_{n+m} + p_{n+m}x_{n+m+k} \leq x_{n+m} + p x_{n+m+k}. \tag{2.34}$$

From the equation (1.2) we have

$$\Delta(r_n \Delta u_n) + q_n x_{n+\ell} + v_n x_{n+m}^{\alpha} = 0, \tag{2.35}$$

and

$$p^{\alpha}\Delta(r_{n+k}\Delta u_{n+k}) + p^{\alpha} q_{n+k} x_{n+\ell+k} + p^{\alpha} v_{n+k} x_{n+m+k}^{\alpha} = 0. \tag{2.36}$$

Combining (2.35) and (2.36), then using (2.29) we get

$$\begin{aligned}\Delta(r_n\Delta u_n + p^{\alpha} r_{n+k}\Delta u_{n+k}) \;&+\; R_n(x_{n+\ell} + p^{\alpha} x_{n+\ell+k}) \\ &+\; S_n(x_{n+m}^{\alpha} + p^{\alpha} x_{n+m+k}^{\alpha}) \leq 0.\end{aligned}$$

Now using (2.33) and (2.34) in above inequality, and also applying Lemmas 2.1 and 2.2, we have

$$\Delta(r_n\Delta u_n + p^{\alpha} r_{n+k}\Delta u_{n+k}) + R_n u_{n+\ell} + T_n u_{n+m}^{\alpha} \leq 0. \tag{2.37}$$

Where $T_n = \begin{cases} S_n & \text{if } 0 < \alpha \leq 1, \\ 2^{1-\alpha} S_n & \text{if } \alpha > 1. \end{cases}$

Summing the inequality (2.37) from n to ∞, we have

$$-(r_n\Delta u_n + p^{\alpha} r_{n+k}\Delta u_{n+k}) + \sum_{s=n}^{\infty} R_s u_{s+\ell} + \sum_{s=n}^{\infty} T_s u_{s+m}^{\alpha} \leq 0$$

$$r_n\Delta u_n + p^{\alpha} r_{n+k}\Delta u_{n+k} \geq \sum_{s=n}^{\infty} R_s u_{s+\ell} + \sum_{s=n}^{\infty} T_s u_{s+m}^{\alpha}$$

$$r_n\Delta u_n(1+p^\alpha) \geq \sum_{s=n}^{\infty} R_s u_{s+\ell} + \sum_{s=n}^{\infty} T_s u_{s+m}^\alpha$$

$$\Delta u_n \geq \frac{1}{1+p^\alpha}\Big(\frac{1}{r_n}\sum_{s=n}^{\infty} R_s u_{s+\ell} + \frac{1}{r_n}\sum_{s=n}^{\infty} T_s u_{s+m}^\alpha\Big).$$

Since $\{u_n\}$ is nondecreasing. Using (2.30) and (2.31) in above inequality, we have

$$\Delta u_n \geq \frac{1}{1+p^\alpha}\Big(R_n^* u_{n+\ell} + S_n^* u_{n+m}^\alpha\Big),$$

or

$$\Delta u_n - \frac{1}{1+p^\alpha}\Big(R_n^* u_{n+\ell} + S_n^* u_{n+m}^\alpha\Big) \geq 0,$$

which is a contradiction for the positivity of $\{u_n\}$. □

Corollary 2.5. *If* $\ell < m,\ \alpha = 1$ *and*

$$\lim_{n\to\infty}\inf \sum_{s=n-m}^{n-1}(R_s^* + S_s^*) > (1+p)\Big(\frac{m}{m+1}\Big)^{m+1} \tag{2.38}$$

then every solution of equation (1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.5 in Theorem 2.2 and the details are omitted. □

Corollary 2.6. *If* $\ell > m,\ \alpha = 1$ *and*

$$\lim_{n\to\infty}\inf \sum_{s=n-\ell}^{n-1}(R_s^* + S_s^*) > (1+p)\Big(\frac{\ell}{\ell+1}\Big)^{\ell+1} \tag{2.39}$$

then every solution of equation (1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.5 in Theorem 2.2 and the details are omitted. □

Corollary 2.7. *Assume* $q_n \equiv 0,\ \ m > k$ *and* $0 < \alpha < 1$ *in equation* (1.2). *If*

$$\sum_{s=n-m+1}^{n-1} S_s^* > 0 \tag{2.40}$$

then every solution of equation (1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.6 in Theorem 2.2 and hence the details are omitted. □

Corollary 2.8. *Assume* $q_n \equiv 0, \quad m > k$ *and* $\alpha > 1$ *in equation* (1.2). *If there exists a* $\lambda > 0$ *such that* $\lambda > \frac{1}{k} log\alpha$ *and*

$$\lim_{s\to\infty} \inf[S_s^* exp(-e^{\lambda s})] > 0, \tag{2.41}$$

then every solution of equation (1.2) *is oscillatory.*

Proof. The proof follows by applying Lemma 2.4 in Theorem 2.2 and hence the details are omitted. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n-1})\Big) + \frac{1}{n}x_{n-2} + \frac{1}{n}x_{n-3} = 0, \quad n \geq 1. \tag{3.1}$$

Here $r_n = \frac{1}{n}, \ p_n = 2, \ q_n = \frac{1}{n}, \ v_n = \frac{1}{n}, \ k = 1, \ \ell = 2, \ m = 3,$ and $\alpha = 1$. It is easy to see that all conditions of Corollary 2.1 are satisfied. Hence every solution of equation (3.1) is oscillatory.

Example 3.2. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n-1})\Big) + \frac{1}{n}x_{n-3} + \frac{1}{n}x_{n-2} = 0, \quad n \geq 1. \tag{3.2}$$

Here $r_n = \frac{1}{n}, \ p_n = 2, \ q_n = \frac{1}{n}, \ v_n = \frac{1}{n}, \ k = 1, \ \ell = 3, \ m = 2,$ and $\alpha = 1$. It is easy to see that all conditions of Corollary 2.2 are satisfied. Hence every solution of equation (3.2) is oscillatory.

Example 3.3. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n-2})\Big) + \frac{1}{n^{7/5}}x_{n-3}^{1/5} = 0, \quad n \geq 1. \tag{3.3}$$

Here $r_n = \frac{1}{n}, \ p_n = 2, \ q_n = 0, \ v_n = \frac{1}{n^{7/5}}, \ k = 2, \ m = 3, \ \alpha = \frac{1}{5}$. It is easy to see that all conditions of Corollary 2.3 are satisfied. Hence every solution of equation (3.3) is oscillatory.

Example 3.4. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n-3})\Big) + \frac{e^{e^n}}{n^{10}} x_{n-4}^5 = 0, \quad n \geq 1. \tag{3.4}$$

Here $r_n = \frac{1}{n}$, $p_n = 3$, $q_n = 0$, $v_n = \frac{e^{e^n}}{n^{10}}$, $k = 2$, $m = 4$, $\alpha = 5$. Choose $\lambda = 1$, then it is easy to see that all conditions of Corollary 2.4 are satisfied. Hence every solution of equation (3.4) is oscillatory.

Example 3.5. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n+1})\Big) + \frac{1}{n(n+1)} x_{n+2} + \frac{1}{n(n+1)} x_{n+3} = 0, \quad n \geq 1. \tag{3.5}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{1}{n(n+1)}$, $v_n = \frac{1}{n(n+1)}$, $k = 1$, $\ell = 2$, $m = 3$, and $\alpha = 1$. It is easy to see that all conditions of Corollary 2.6 are satisfied. Hence every solution of equation (3.5) is oscillatory.

Example 3.6. Consider the neutral difference equation

$$\Delta\Big(\frac{1}{n}\Delta(x_n + 2x_{n+1})\Big) + \frac{1}{n(n+1)} x_{n+3} + \frac{1}{n(n+1)} x_{n+2} = 0, \quad n \geq 1. \tag{3.6}$$

Here $r_n = \frac{1}{n}$, $p_n = 2$, $q_n = \frac{1}{n(n+1)}$, $v_n = \frac{1}{n(n+1)}$, $k = 1$, $\ell = 3$, $m = 2$, and $\alpha = 1$. It is easy to see that all conditions of Corollary 2.6 are satisfied. Hence every solution of equation (3.6) is oscillatory.

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