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NINTH ORDER BLOCK PIECEWISE CONTINUOUS HYBRID INTEGRATORS FOR SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: This paper presents a Piecewise Continuous Hybrid Integrator (PCHI) of order nine that has the ability to produce continuous solution defined for all values of the required independent variable. The PCHI is used to generate several implicit hybrid integrators that are expressed in block form and are applied as a Block Hybrid Integrators (BHI) for the numerical solution of second order initial value problems at both grid and off-grid points. The stability properties of the method are discussed.

Numerical examples are given to illustrate the accuracy and efficiency of the proposed BHI.

AMS Subject Classification: 65L05, 65L06
Key Words: block hybrid integrators, off-step points, stability

1. Introduction

Consider the general second order IVPs of the form

$$y'' = f(t, y, y'),\ y(t_0) = y_0\ ,\ y'(t_0) = y'_0. \tag{1}$$

Equation (1) occurs in several areas of engineering and science and social sciences. It is well known that many of these problems have proved to be either difficult to solve or cannot be solved when their solution are sought for analyt-

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ically, hence the necessity of numerical techniques for such problems remains vital. In practice, problem (1) is solved by first reducing it to a system of first-order differential equations and then applying the various methods available for solving systems of first order IVPs (see Lambert [20], Hairer and Wanner [13]). However, if y' from the right hand side of (1) does not appear explicitly, solving (1) without first reducing it into a system of first order ODEs is preferable, since about half of the storages space can be saved, especially if the dimension of (1) is large(see Hairer, Nrsett and Wanner[12]). Efficient solution of (1) without y' on the right hand side has been demonstrated in Carpentieri and Paternoster [4], Coleman and Duxbury [8], Hairer and Wanner [15], Ixaru and Berghe [6], Simos [23], and Tsitouras [25], Sommeijer [24],and Twizell and Khaliq [26]).

In order to improve storage space and reduce computational time, many authors have solved equation (1) directly without first reducing it to an equivalent first-order system. (see Mahmoud and Osman [21], Hairer and Wanner [14], Chawla and Sharma [5], Vigo-Aguiar and Ramos [27], and Vigo-Aguiar and Ramos [28]). Most of these methods are implemented in a step-by-step fashion in which on the partition Γ, an approximation is obtained at t_{n+1} only after an approximation at t_n has been computed, where $\Gamma : a = t_0 < t_1 < ... < t_N = b$, $t_{n+1} = t_n + h$ $n = 0, 1, ..., N-1$ $h = \frac{b-a}{N}$ is the constant step-size of the partition of Γ N is a positive integer, and n is the grid index. It should be noted that the main hybrid method is given in the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=1}^{v} \beta\eta_j f_{n+\eta_j}, \tag{2}$$

where $\alpha_k = 1$ and $\beta_k \neq 0, k = 4$ is the step number, β_0 and α_0 do not vanish, α_j, β_j, $\beta\eta_j$ are unknown constants, $v = 4$ is the number of off-step points and η_j are rational numbers and cannot be integers. It should be noted that the concept of combining different methods for solving scalar and systems of first order ODEs is due to Donelson III and Hansen [7], Brugnano and Trigiante [3], and Onumanyi et al. [22]. Recently, Akinfenwa et.al.[1] and Jator [17, 18] proposed self-starting linear multistep methods that were used to solve (1) through a block-by-block approach. Following Akinfenwa et.al.[1], a hybrid block integrator is proposed which discretizes the problem using the main and additional methods and simultaneously solves the resulting system in a block-by-block fashion.

The rest of this paper is presented as follows: In Section 2, we discuss the basic idea behind the algorithm and obtain a continuous representation $Y(t)$ for the exact solution $y(t)$ which is used to generate members of the block method for solving (1). In Section 3, we present the analysis of BHI. The implementation of the method is discussed in Section 4. In Section 5,we show the accuracy of the method. Finally, in Section 6 we present some concluding remarks.

2. Derivation of the Method

In this section, a k-step hybrid method is developed for (1) on the interval from $t_n$to t_{n+k}. The initial assumption is that the solution on the interval $[t_n, t_{n+4}]$ is locally approximated by the polynomial

$$Y(t) = \sum_{j=0}^{2k+2} b_j t^j, \tag{3}$$

where b_j are unknown coefficients. Since this polynomial must pass through the interpolation points (t_n, y_n), (t_{n+1}, y_{n+1}) and the collocation points

$$(t_n, y_n, t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}), \ldots (t_{n+k}, y_{n+k}),$$

we demand that the following $(2k + 3)$ equations must be satisfied.

$$Y(t_{n+j}) = y_{n+j}\ , \qquad j = 0, 1. \tag{4}$$

$$Y''(t_{n+\frac{i}{2}}) = f_{n+\frac{i}{2}},\ i = 0, \ldots 2k. \tag{5}$$

The $(2k+3)$ undetermined coefficients b_j are obtained by solving equations (4) and (5) and are then substituted into (3). After some algebraic computation the continuous representation of the hybrid integrator is obtained and given in the form

$$Y(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^2 \sum_{j=1}^{v} \beta\eta_j(t) f_{n+\eta_j} \tag{6}$$

where $\alpha_j(t), j = 0, 1,\ \beta_j(t)$, and $\beta\eta_j(t)$, $j = 0, 1, \ldots, k$ are continuous coefficients k is the step number, $v = 4$ is the number of off-step, and h is the chosen step-length. We assume that $y_{n+j} = Y(t_n + jh)$ is the numerical approximation to the analytical solution $y(t_{n+j}), y'_{n+j} = Y'(t_n + jh)$ is an approximation to $y'(t_{n+j}), f_{n+j} = Y''(t_n + jh)$ is an approximation to $y''(t_{n+j})$, where

$f_{n+\frac{i}{2}} = f(t_{n+\frac{i}{2}}, y_{n+\frac{i}{2}}, y'_{n+\frac{i}{2}}),\ i = 0, \ldots 2k.$

The methods (6) and (7) are used to obtain the main and additional integrators which are combined to provide a global solution for (1)on Γ.

Main Methods. The main methods of the form (2) are obtained by evaluating (6) at $t = t_{n+\frac{i}{2}}, \quad i = 1, \dots 2k$, but $i \neq 2$. Thus, $y_{n+\frac{i}{2}} = Y(t_n + \frac{i}{2}h), i = 1, \dots 2k$, but $i \neq 2, \quad n = 0, k, 2k, \dots, N-k$.

It should be noted that the discretization of (1)using the main methods obtained from (6) give more unknowns than equations which if solved will lead to an indeterminate. Hence, the compulsion to look for additional methods. Interestingly, (6) is continuous and is used to provide the needed methods via its first derivative (7).

$$Y'(t) = \frac{d}{dt}\{\sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^2 \sum_{j=1}^{v} \beta\eta_j(t) f_{n+\eta_j}\} \qquad (7)$$

Additional Methods. Noting that $y'_{n+\frac{i}{2}} = Y'(t_n + \frac{i}{2}h), i = 0, 1, \dots, 2k$, the additional methods are obtained from (7) at $t = t_{n+\frac{i}{2}}, i = 0, 1, \dots, 2k$. $n = 0, k, 2k, \dots, N-k$.

In what follows, are the members of the discrete form taking $k = 4$ and $\eta_{\frac{i}{2}} = (\frac{i}{2})$, $i = 1, 3, 5, 7$.

Discrete Form. The discrete form of the main BHI is obtained by evaluating (6) at $\{t = t_{n+\frac{i}{2}}, \quad i = 1, \dots 2k\}$ but $i \neq 2$. Specifically, the evaluations are performed for $y_{n+\frac{i}{2}} = Y(t_n + \frac{i}{2}h), \quad i = 1, \dots 2k$ but $i \neq 2$, to yield (8). While the additional methods obtained from (7) at $\{t = t_{n+\frac{i}{2}}, \quad i = 0, 1, \dots 2k\}$ are performed for $hy'_{n+\frac{i}{2}} = Y'(t_n + \frac{i}{2}h), \quad i = 0, 1, \dots 2k$ to yield (9).

$$
\begin{cases}
y_{n+\frac{1}{2}} = \frac{y_n}{2} + \frac{y_{n+1}}{2} + \frac{h^2}{29030400}(-237671f_n - 3398072f_{n+\frac{1}{2}} + 653032f_{n+1} \\
-1426304f_{n+\frac{3}{2}} + 1376650f_{n+2} \\
-884504f_{n+\frac{5}{2}} + 368272f_{n+3} - 90032f_{n+\frac{7}{2}} + 9829f_{n+4}) \\
y_{n+\frac{3}{2}} = -\frac{y_n}{2} + \frac{3y_{n+1}}{2} + \frac{h^2}{9676800}(72671f_n + 1350112f_{n+\frac{1}{2}} + 1811808f_{n+1} \\
+590504f_{n+\frac{3}{2}} - 333650f_{n+2} \\
+202704f_{n+\frac{5}{2}} - 83512f_{n+3} + 20392f_{n+\frac{7}{2}} - 2229f_{n+4}) \\
y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{226800}(3431f_n + 62912f_{n+\frac{1}{2}} + 90908f_{n+1} \\
+73184f_{n+\frac{3}{2}} - 9850f_{n+2} \\
+9344f_{n+\frac{5}{2}} - 4012f_{n+3} + 992f_{n+\frac{7}{2}} - 109f_{n+4}) \\
y_{n+\frac{5}{2}} = -\frac{3y_n}{2} + \frac{5y_{n+1}}{2} + \frac{h^2}{1935360}(43983f_n + 804232f_{n+\frac{1}{2}} + 1184552f_{n+1} \\
+1185168f_{n+\frac{3}{2}} + 282070f_{n+2} \\
+173192f_{n+\frac{5}{2}} - 56352f_{n+3} + 13408f_{n+\frac{7}{2}} - 1453f_{n+4}) \\
y_{n+3} = -2y_n + 3y_{n+1} + \frac{h^2}{75600}(2299f_n + 41872f_{n+\frac{1}{2}} + 62208f_{n+1} \\
+68144f_{n+\frac{3}{2}} + 27250f_{n+2} \\
+25584f_{n+\frac{5}{2}} - 1072f_{n+3} + 592f_{n+\frac{7}{2}} - 69f_{n+4}) \\
y_{n+\frac{7}{2}} = -\frac{5y_n}{2} + \frac{7y_{n+1}}{2} + \frac{h^2}{829440}(31511f_n + 573296f_{n+\frac{1}{2}} + 860720f_{n+1} \\
+979448f_{n+\frac{3}{2}} + 487790f_{n+2} \\
+497024f_{n+\frac{5}{2}} + 174584f_{n+3} + 25880f_{n+\frac{7}{2}} - 1453f_{n+4}) \\
y_{n+4} = -3y_n + 4y_{n+1} + \frac{h^2}{37800}(1701f_n + 31552f_{n+\frac{1}{2}} + 46388f_{n+1} \\
+57504f_{n+\frac{3}{2}} + 27250f_{n+2} \\
+36224f_{n+\frac{5}{2}} + 14748f_{n+3} + 10912f_{n+\frac{7}{2}} + 521f_{n+4})
\end{cases}
\tag{8}
$$

where $n = 0, k, \ldots, N-k$.

$$
\begin{cases}
hy'_n = -y_n + y_{n+1} + \frac{h^2}{453600}(-58193f_n - 235072f_{n+\frac{1}{2}} + 183708f_{n+1} \\
-247328f_{n+\frac{3}{2}} \\
+227030f_{n+2} - 143232f_{n+\frac{5}{2}} + 59092f_{n+3} - 14368f_{n+\frac{7}{2}} + 1563f_{n+4}) \\
hy'_{n+\frac{1}{2}} = -y_n + y_{n+1} + \frac{h^2}{7257600}(138929f_n + 705942f_{n+\frac{1}{2}} \\
-1665266f_{n+1} + 1638110f_{n+\frac{3}{2}} - 1400640f_{n+2} \\
+854626f_{n+\frac{5}{2}} - 345742f_{n+3} + 82986f_{n+\frac{7}{2}} - 8945f_{n+4}) \\
hy'_{n+1} = -y_n + y_{n+1} + \frac{h^2}{453600}(6561f_n + 130096f_{n+\frac{1}{2}} + 98720f_{n+1} \\
-7152f_{n+\frac{3}{2}} - 5210f_{n+2} \\
+6224f_{n+\frac{5}{2}} - 3216f_{n+3} + 880f_{n+\frac{7}{2}} - 103f_{n+4}) \\
hy'_{n+\frac{3}{2}} = -y_n + y_{n+1} + \frac{h^2}{7257600}(112273f_n + 1981910f_{n+\frac{1}{2}} \\
+3217806f_{n+1} + 2517406f_{n+\frac{3}{2}} - 916480f_{n+2} \\
+497442f_{n+\frac{5}{2}} - 193550f_{n+3} + 45674f_{n+\frac{7}{2}} - 4881f_{n+4}) \\
hy'_{n+2} = -y_n + y_{n+1} + \frac{h^2}{453600}(6815f_n + 126144f_{n+\frac{1}{2}} + 187612f_{n+1} \\
+27670f_{n+\frac{3}{2}} + 81750f_{n+2} \\
+4480f_{n+\frac{5}{2}} - 4204f_{n+3} + 1248f_{n+\frac{7}{2}} - 149f_{n+4}) \\
hy'_{n+\frac{5}{2}} = -y_n + y_{n+1} + \frac{h^2}{7257600}(111537f_n + 1992598f_{n+\frac{1}{2}} \\
+3128078f_{n+1} + 4001502f_{n+\frac{3}{2}} + 3532480f_{n+2} \\
+1981538f_{n+\frac{5}{2}} - 283278f_{n+3} + 56362f_{n+\frac{7}{2}} - 5617f_{n+4}) \\
hy'_{n+3} = -y_n + y_{n+1} + \frac{h^2}{453600}(6769f_n + 126512f_{n+\frac{1}{2}} + 186624f_{n+1} \\
+274960f_{n+\frac{3}{2}} + 168710f_{n+2} \\
+288336f_{n+\frac{5}{2}} + 84688f_{n+3} - 2704f_{n+\frac{7}{2}} + 105f_{n+4}) \\
hy'_{n+\frac{7}{2}} = -y_n + y_{n+1} + \frac{h^2}{7257600}(115601f_n + 1955286f_{n+\frac{1}{2}} \\
+3280270f_{n+1} + 3644318f_{n+\frac{3}{2}} + 4016640f_{n+2} \\
+2860834f_{n+\frac{5}{2}} + 4599794f_{n+3} + 1332330f_{n+\frac{7}{2}} - 32273f_{n+4}) \\
hy'_{n+4} = -y_n + y_{n+1} + \frac{h^2}{453600}(5103f_n + 141760f_{n+\frac{1}{2}} + 124316f_{n+1} \\
+424416f_{n+\frac{3}{2}} - 63530f_{n+2} \\
+528512f_{n+\frac{5}{2}} - 300f_{n+3} + 362464f_{n+\frac{7}{2}} + 64859f_{n+4})
\end{cases}
\tag{9}
$$

where $n = 0, k, \dots, N-k$.

The advantage of implementing (8) and (9) simultaneously is that the number of function evaluations are unchanged. Although, it require solving a system with twice the number of variables.

3. Analysis of the BHI

In this section, we discuss the zero-stability, local truncation error and order, consistency, and convergence of the BHI.

Zero-Stability. The methods displayed in (8) together with the first member of (9) which were used as an auxiliary method to solve (1) and can be represented by a matrix finite difference equation in block form given by

$$A^{(1)}Y_{\varpi+1} = A^{(0)}Y_{\varpi} + h^2[B^{(1)}F_{\varpi+1} + B^{(0)}F_{\varpi}] + hC^{(0)}\Delta_{\varpi}, \tag{10}$$

where

$$Y_{\varpi+1} = (y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4})^T,$$

$$Y_{\varpi} = (y_{n-\frac{7}{2}}, y_{n-3}, y_{n-\frac{5}{2}}, y_{n-2}, y_{n-\frac{3}{2}} y_{n-1}, y_{n-\frac{1}{2}}, y_n)^T,$$

$$F_{\varpi+1} = (f_{n+\frac{1}{2}}, f_{n+1}, y_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}, f_{n+\frac{7}{2}}, f_{n+4})^T,$$

$$F_{\varpi} = (f_{n-\frac{7}{2}}, f_{n-3}, f_{n-\frac{5}{2}}, f_{n-2}, f_{n-\frac{3}{2}} f_{n-1}, f_{n-\frac{1}{2}}, f_n)^T,$$

$$\Delta_{\varpi} = (\zeta_{n-\frac{7}{2}}, \zeta_{n-3}, \zeta_{n-\frac{5}{2}}, \zeta_{n-2}, \zeta_{n-\frac{3}{2}}, \zeta_{n-1}, \zeta_{n-\frac{1}{2}}, \zeta_n)^T,$$

$\varpi = 0, 1, 2, \ldots$, ζ is the derivative of y, and $n = 0, 4, \ldots$ and the matrices $A^{(1)}, A^{(0)}, B^{(1)}, B^{(0)}$ and $C^{(0)}$ are defined as follow

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} -\frac{424759}{3628800} & \frac{81629}{3628800} & -\frac{11143}{226800} & \frac{27533}{580608} & -\frac{110563}{3628800} & \frac{23017}{1814400} & -\frac{5627}{1814400} & -\frac{9829}{29030400} \\ -\frac{7346}{14175} & \frac{81}{200} & -\frac{7729}{14175} & \frac{22703}{45360} & -\frac{1492}{4725} & \frac{14773}{113400} & -\frac{449}{14175} & \frac{521}{151200} \\ \frac{42191}{302400} & \frac{2097}{11200} & \frac{73813}{1209600} & -\frac{6673}{193536} & \frac{4223}{201600} & -\frac{10439}{1209600} & \frac{2549}{1209600} & -\frac{743}{3225600} \\ \frac{3932}{14175} & \frac{22727}{56700} & \frac{4574}{14175} & -\frac{197}{4536} & \frac{584}{14175} & -\frac{1003}{56700} & \frac{62}{14175} & \frac{109}{226800} \\ \frac{100529}{241920} & \frac{148069}{241920} & \frac{24691}{40320} & \frac{28207}{193536} & \frac{21649}{241920} & -\frac{587}{20160} & \frac{419}{60480} & -\frac{1453}{1935360} \\ \frac{2617}{4725} & \frac{144}{175} & \frac{4259}{4725} & \frac{545}{1512} & \frac{533}{1575} & -\frac{67}{4725} & \frac{37}{4725} & -\frac{23}{25200} \\ \frac{35831}{51840} & \frac{10759}{10368} & \frac{122431}{103680} & \frac{48779}{82944} & \frac{3883}{6480} & \frac{21823}{103680} & \frac{647}{20736} & -\frac{1453}{829440} \\ \frac{3944}{4725} & \frac{11597}{9450} & \frac{2396}{1575} & \frac{545}{756} & \frac{4528}{4725} & \frac{1229}{3150} & \frac{1364}{4725} & \frac{521}{37800} \end{pmatrix},$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{33953}{4147200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{58193}{453600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{72671}{9676800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3431}{226800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1629}{71680} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2291}{75600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{31511}{829440} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{200} \end{pmatrix},$$

$$C^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The zero stability of the BHI determined as the limit h tends to zero. Thus as $h \to 0$ the method (10) tends to the difference system

$$A^{(1)} Y_{\varpi+1} = A^{(0)} Y_{\varpi}$$

which is normalized to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R\widehat{A^{(1)}} - \widehat{A^{(0)}}) = R^7(R-1) \tag{11}$$

where $\widehat{A^{(1)}}$ is an identity matrix of dimension 8 and $\widehat{A^{(0)}}$ is given by

$$\widehat{A^{(0)}} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

Following Fatunla [9], the block method (10) is zero-stable, since from (11), $\rho(R) = 0$ satisfies $|R_j| \leq= 1$, $j = 1, \ldots, 8$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2.We note that the single members of the block method are not zero-stable, but this property is gained when the methods are combined as numerical integrators in the block form (10).

Local Truncation Error and Order. Following Fatunla [9] and Lambert [19] we define the local truncation error associated with (10) to be the linear difference operator

$$L[y(t); h] = \sum_{j=0}^{k} \{\alpha_j y(t + jh) - h^2 \beta_j y''(t + jh)\} - h^2 \sum_{j=1}^{v} \beta\eta_j y''(t + \eta_j h) \quad (12)$$

Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (12) as a Taylor series about the point t to obtain the expression

$$L[y(t); h] = C_0 y(t) + C_1 y'(t) + \ldots + C_s h^s y^(s)(t) + \ldots,$$

where the constant coefficients $C_s, s = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j,$$

$$\vdots$$

$$C_s = \frac{1}{s!}\sum_{j=0}^{k} j^s \alpha_j - \frac{1}{(s-2)!}(\sum_{j=0}^{k} j^{s-2}\beta_j + \sum_{j=0}^{k} \eta_j^{s-2}\beta\eta_j).$$

According to [16], we say that the method (10) has order s if

$$C_0 = C_1 = \ldots = C_s = C_{s+1} = 0, \quad C_{s+2} \neq 0.$$

Therefore, C_{s+2} is the error constant and $C_{s+2}h^{s+2}y^{(s+2)}(t_n)$ the principal local truncation error at the point t_n.

Thus, we can write the local truncation error (LTE) of the method of order s as

$$LTE = C_{s+2}h^{s+2}y^{(s+2)}(t_n) + \bigcirc(h^{s+3}).$$

It is established from our calculations that the block hybrid integrators given in (10) have order s and error constant C_9 given by the vectors $s = (9, 9, 9, 9, 9, 9, 9, 9)^T$ and

$$C_9 = (\frac{407}{707788800}, \frac{22063}{3832012800}, -\frac{5983}{14863564800}, -\frac{7}{8294400}, -\frac{3821}{2972712960}, -\frac{7}{4147200}, -\frac{961}{424673280}, -\frac{7}{4147200})^T.$$

Consistency and Convergence. The block method (10) is consistent since each of the integrators has order $s > 1$. According to Henrici [16], convergence = consistency + zero-stability. Hence the BHI is convergent.

Stability Analysis. The stability property of (10) is discussed in the spirit [13] by applying it to the test equation

$$y'' = -\lambda^2 y, \quad \lambda \in R$$

to yield

$$Y_{\varpi+1} = D(z)Y_{\varpi}\ , \ z = \lambda h,$$

where the matrix $D(z)$ is given by

$$[D(z) = (A^{(1)} + z^2B^{(1)})^{-1}(A^{(0)} - z^2B^{(0)} + izC^{(0)})]$$

is the amplification matrix.

Definition 3.1. The interval $(-z_0, 0)$ is a stability interval for the block method (10), if in this interval $\rho(z) < 1$, where $\rho(z)$ is the spectral radius of $D(z)$ (see [2]).

We found that $\rho(z) \leq 1$ if $z \in (-16.74, 0)$. Thus the BHI has a moderately wide interval of stability.

4. Implementation

The implementation of the above block methods is summarized as follows: On the partition

$$I_N : \{a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b\}, \quad n = 0, 1, 2, \ldots, N-1.$$

Step 1. Choose N, for $k = 4, h = (b-a)/N$, the number of blocks $\Gamma = N/4$. Using (11), $n = 0, \varpi = 0$, the values of $(y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2, y_{\frac{5}{2}}, y_3, y_{\frac{7}{2}}, y_4)^T$ are simultaneously obtained over the sub-interval $[t_0, t_{\frac{i}{2}}]$, $i = 1, 2...8$, as y_0 is known from the IVPs (1).

Step 2: For $n = 4, \varpi = 1$, the values of $(y_{\frac{9}{2}}, y_5, y_{\frac{11}{2}}, y_6, y_{\frac{13}{2}}, y_7, y_{\frac{15}{2}}, y_8)^T$ are simultaneously obtained over the sub-interval $[t_4, t_{\frac{i}{2}}]$, $i = 9, 10...16$, as y_4 is known form the previous block.

Step 3: The process is continued for $n = 8, \ldots, N-2$ and $\varpi = 2, \ldots, \Gamma$ to obtain approximate solutions to (1) on sub-intervals $[t_8, t_{12}], \ldots, [t_{N-4}, t_N]$.

Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional fashion.

We explain briefly the implementation of the block method. For linear problem we use the Gaussian elimination to solve the resulting 2k x2k matrix in each block with our written Matlab code. While for non- linear problems the code uses the Newton iteration. The following notation is used to specify the iteration $y^{j+1}_{n+\frac{i}{2}}$ denotes the $(j+1)th$ iterative value of $y_{n+\frac{i}{2}}$ and $\delta^{j+1}_{n+i} = y^{j+1}_{n+i} - y^{j}_{n+i}$ for $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots$. Thus the Newton iteration of the hybrid block integrator (10) takes the form

$$y^{(j+1)}_{n+\frac{i}{2}} = y^{(j)}_{n+\frac{i}{2}} - \frac{f^{(j)}_{n+\frac{i}{2}}}{f'^{(j)}_{n+\frac{i}{2}}}, \quad i = 1, 2, \ldots, 2k, \tag{13}$$

$$y^{(j+1)}_{n+\frac{1}{2}} - y^{(j)}_{n+\frac{1}{2}} = \frac{a_1 y^{(j)}_{n+1} + h^2\eta_1 f^{(j)}_{n+\frac{1}{2}} + h^2\beta_1 f^{(j)}_{n+1} \ldots + h^2\eta_4 f^{(j)}_{n+\frac{7}{2}} + h^2\beta_4 f^{(j)}_{n+4}}{1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\eta_4 \frac{\delta f_{n+\frac{7}{2}}}{\delta y_{n+\frac{7}{2}}} + h^2\beta_4 \frac{\delta f_{n+4}}{\delta y_{n+4}}} + D_1,$$

$$y^{(j+1)}_{n+1} - y^{(j)}_{n+1} = \frac{h^2\theta_1 f^{(j)}_{n+\frac{1}{2}} + h^2\nu_1 f^{(j)}_{n+1} \ldots + h^2\theta_4 f^{(j)}_{n+\frac{7}{2}} + h^2\nu_4 f^{(j)}_{n+4}}{1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\theta_4 \frac{\delta f_{n+\frac{7}{2}}}{\delta y_{n+\frac{7}{2}}} + h^2\nu_4 \frac{\delta f_{n+4}}{\delta y_{n+4}}} + D_2,$$

$$y^{(j+1)}_{n+\frac{3}{2}} - y^{(j)}_{n+\frac{3}{2}} = \frac{c_1 y^{(j)}_{n+1} + h^2\vartheta_1 f^{(j)}_{n+\frac{1}{2}} + h^2\mu_1 f^{(j)}_{n+1} \ldots + h^2\vartheta_4 f^{(j)}_{n+\frac{7}{2}} + h^2\mu_4 f^{(j)}_{n+4}}{1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\vartheta_4 \frac{\delta f_{n+\frac{7}{2}}}{\delta y_{n+\frac{7}{2}}} + h^2\mu_4 \frac{\delta f_{n+4}}{\delta y_{n+4}}} + D_3,$$

$$\ldots$$

$$y^{(j+1)}_{n+4} - y^{(j)}_{n+4} = \frac{g_1 y^{(j)}_{n+1} + h^2\tau_1 f^{(j)}_{n+\frac{1}{2}} + h^2\kappa_1 f^{(j)}_{n+1} \ldots + h^2\tau_4 f^{(j)}_{n+\frac{7}{2}} + h^2\kappa_4 f^{(j)}_{n+4}}{1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\tau_4 \frac{\delta f_{n+\frac{7}{2}}}{\delta y_{n+\frac{7}{2}}} + h^2\kappa_4 \frac{\delta f_{n+4}}{\delta y_{n+4}}} + D_8.$$

We can express the procedure in matrix form as follows:

$$J^{(1)}\delta^{(1)} = \alpha^{(0)} Y^{(1)} + h\beta^{(0)} F^{(1)} + D. \tag{14}$$

Here

$$J^{(1)} = \begin{pmatrix} 1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\beta_4 \frac{\delta f_{n+4}}{\delta y_{n+4}} & a_1 & 0 & \ldots & 0 \\ 0 & 1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\nu_4 \frac{\delta f_{n+4}}{\delta y_{n+4}} & 0 & \ldots & 0 \\ \vdots & \cdots & \vdots & & \\ 0 & g_1 & 0 & \ldots & 1 + h\frac{\delta f_{n+\frac{1}{2}}}{\delta y_{n+\frac{1}{2}}} + \ldots + h^2\kappa_4 \frac{\delta f_{n+4}}{\delta y_{n+4}} \end{pmatrix},$$

$$\alpha^{(0)} = \begin{pmatrix} -1 & -a_1 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -e_1 & \ldots & -1 & 0 \\ 0 & -g_1 & 0 & \ldots & -1 \end{pmatrix},$$

$$Y^{(1)} = (y^{(j)}_{n+\frac{1}{2}}, y^{(j)}_{n+1}, y^{(j)}_{n+\frac{3}{2}}, \ldots, y^{(j)}_{n+\frac{7}{2}}, y^{(j)}_{n+4})^T,$$

$$\beta^{(0)} = \begin{pmatrix} \eta_1 & \beta_1 & \eta_2 & \ldots & \beta_4 \\ \theta_1 & \nu_1 & \theta_2 & \ldots & \nu_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_1 & \kappa_1 & \tau_2 & \ldots & \kappa_4 \end{pmatrix},$$

$$F^{(1)} = (f^{(j)}_{n+\frac{1}{2}}, f^{(j)}_{n+1}, f^{(j)}_{n+\frac{3}{2}}, \ldots, f^{(j)}_{n+\frac{7}{2}}, f^{(j)}_{n+4})^T,$$

$$D = (D_1, D_2, D_3, \ldots, D_7, D_8)^T$$

$D_1, D_2, \ldots, D_8$ are known from the initial value of the problem and T is the transpose. Thus we obtain the approximated values of $y_{n+\frac{1}{2}}, y_{n+1}, \ldots, y_{n+4}$ as

$$y^{(j+1)}_{n+\frac{1}{2}} = y^{(j)}_{n+\frac{1}{2}} + \delta^{(j+1)}_{n+\frac{1}{2}},$$

$$y^{(j+1)}_{n+1} = y^{(j)}_{n+1} + \delta^{(j+1)}_{n+1},$$

$$\vdots$$

$$y^{(j+1)}_{n+4} = y^{(j)}_{n+4} + \delta^{(j+1)}_{n+4}.$$

5. Numerical Examples

Example 5.1. We consider the given nonlinear Duffing equation which was also solved by Tsitouras [25] on $[0, \frac{20.5}{1.01}\pi]$

$$y'' = -y - y^3 + C_0 \cos(1.01),\ y(0) = 0.200426728069\ y'(0) = 0$$

$$y(t) = C_1 \cos(1.01t) + C_2 \cos(3.03t) + C_3 \cos(5.05t) + C_4 \cos(7.07t)$$

where $C_0 = 0.002; C_1 = 0.200179477536; C_2 = 0.246946143 \times 10^{-3}; C_3 = 0.304016 \times 10^{-6}; C_4 = 0.374 \times 10^{-9}$.

The maximum norm of the global error is given in the form 10^{CD}, where CD denotes the number correct decimal digits at the endpoint (see [24]). This problem was solved in [25] using the eighth order method of Tsitouras (PL22), and Tsitouras (NEWTSI) in [25]. The results obtained using the method in [25] are reproduced in Table 1 and compared with the results obtained by the BHI. As expected, it is seen from Table 1 that BHI performs better than both method of Tsitouras in [25] in terms of accuracy (smaller errors) and efficiency (smaller number of function evaluations).

Example 5.2. We consider the Bessels ODE (see Vigo et.al.[28] and Jator[17]) given by

$$y'' = -\frac{y'}{t} - (1 - \frac{y}{4t^2}),\ y(1) = 2\sqrt{\frac{2}{\pi}} \sin 1 \simeq 0.6713967061418031$$

$$y'(1) = (2\cos 1 - \sin 1)/\sqrt{2\pi} \simeq 0.0954005144474746.$$

Table 1: The correct decimal digit at the endpoint for Example 5.1

Tsitouras(PL22)		Tsitouras(NEWTSI)	BHI	
NFEs	CD	CD	NFEs	CD
2000	4.5	5.7	201	6.2
3000	6.9	8.2	401	8.4
4000	8.3	9.6	697	10.6
5000	9.3	10.5	1393	11.7
6000	10.1	10.9	1201	11.3
7000	10.9	11.1	2193	11.9

Its exact solution is given by

$$y(t) = \jmath_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \sin t.$$

The theoretical solution at $t = 8$ is $y(8) = \sqrt{\frac{2}{8\pi}} \sin 8 \simeq 0.279092789108058969$. The absolute errors for the y-component and its derivative were obtained at $t = 8$ using the BHI for fixed step-sizes $h = 7/8, 7/16, 7/32, 7/64, 7/128$ corresponding to the number of steps $N = 8, 16, 32, 64, 128$ as shown in Table 2. Similar results were obtained for the same problem in [28] using the variable-step Falker method of order eight (p = 8) implemented in the predictorcorrector mode(PC) and[17] using hybrid third derivative method of order (p=8). As expected, it is seen generally that the BHI perform better than those in [28]and [17].

Table 2: Absolute errors=$|y(t) - y|$ for Example 5.2

	Vigo et.al.[28]			BHTDA			BHI	
N	errory(y(t))	error(y'(t))	N	errory(y(t))	error(y'(t))	N	errory(y(t))	error(y'(t))
67	7.1122×10^{-7}	6.0624×10^{-7}	8	9.9694×10^{-6}	2.5337×10^{-5}	8	2.1636×10^{-5}	3.4077×10^{-5}
82	9.2632×10^{-8}	4.0342×10^{-7}	16	2.3496×10^{-7}	3.9066×10^{-7}	16	6.9641×10^{-7}	2.1337×10^{-7}
97	8.7834×10^{-9}	3.6054×10^{-8}	32	2.2325×10^{-9}	3.1724×10^{-9}	32	4.1246×10^{-9}	1.7134×10^{-9}
112	1.2108×10^{-10}	8.2920×10^{-9}	64	1.2356×10^{-11}	1.7059×10^{-11}	64	9.6898×10^{-12}	1.8506×10^{-12}
125	2.7068×10^{-11}	1.0044×10^{-11}	128	5.9008×10^{-14}	7.8278×10^{-14}	128	1.2934×10^{-14}	5.6968×10^{-15}

Example 5.3. We consider the nonlinear perturbed system on the range $[0, 10]$ with $\epsilon = 10^{-3}$

$$y_1'' + 25y_1 + \epsilon(y_1^2 + y_2^2) = \epsilon\varphi_1(t), y_1(0) = 1, y_1'(0) = 0.$$
$$y_2'' + 25y_2 + \epsilon(y_1^2 + y_2^2) = \epsilon\varphi_2(t), y_2(0) = \epsilon, y_2'(0) = 5.$$

Where

$$\varphi_1(t) = 1 + \epsilon^2 + 2\epsilon\sin(5t + t^2) + 2\cos(t^2) + (25 - 4t^2)\sin(t^2)$$

$$\varphi_2(t) = 1 + \epsilon^2 + 2\epsilon\sin(5t + t^2) - 2\sin(t^2) + (25 - 4t^2)\cos(t^2)$$

and the exact solution is given by

$$y_1 t = \cos(5t) + \epsilon\sin(t^2),$$
$$y_2 t = \sin(5t) + \epsilon\cos(t^2),$$

represents a periodic motion of constant frequency with small perturbation of variable frequency. This problem was chosen to demonstrate the performance of the BHI on a nonlinear perturbed system. The problem was also solved by Fang et al. [10] using a variable step-size fifth-order trigonometrically fitted RungeKutta-Nystrm method TFRKN5 and a fifth-order RungeKuttaNystrm method (RKN5) which was constructed by Franco [11].In Table 2, the maximum global error ($Err = Max|y(t) - y|$) for the three methods are compared. We remark that the TFARKN5 and ARKN5 are expected to perform better because they are exact when the solution involves a linear combination of trigonometric functions as well as implemented as a variable-step method. Nevertheless, the BHI which is implemented using a fixed step-size is highly competitive to them, especially as the step-size is decreased. We have also calculated the rate of convergence (ROC) of the BHI using the formula $ROC = log_2 E^{2h}/E^h$, where $E^h = Err$ using the step size h. We show that the calculated ROC of the BHI is consistent with the theoretical order ($p = 9$) behavior of the method, since on halving the step size, the error is reduced by a factor of about 2^8.

Example 5.4. We consider the nonlinear Fehlberg problem which was also solved in Sommeijer [24] and Jator [17].

$$y_1'' = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}},$$

$$y_2'' = \frac{2y_1}{\sqrt{y_1^2 + y_2^2}} - 4t^2 y_2$$

$$y_1(0) = 0, y_1'(0) = -2\sqrt{\tfrac{\pi}{2}}, \quad y_2(0) = 1, y_2'(0) = 0$$
$$y_1(t) = \cos(x^2), \quad y_2(t) = \sin(t^2)$$

Table 3: Absolute errors,$|y(t) - y|$ for Example 5.3

RKN5		TFRKN5		BHI		
N(Rejected)	$-log_{10}$(Err)	N(Rejected)	$-log_{10}$(Err)	N	$-log_{10}$(Err)	ROC
42(15)	2.82	29(6)	2.78	50	2.2	-
86(7)	4.96	88(9)	5.33	100	5.18	9.9
260(5)	7.16	262(8)	7.85	200	7.90	9.0
812(3)	9.37	811(4)	10.38	400	11.28	10.7

Table 4: The correct decimal digit at the endpoint for Example 5.4

PIRKN III		H8	H10	BHTDA		BHI	
NFEs	CD	CD	CD	NFEs	CD	NFEs	CD
400	2.7	0.3	-0.3	398	5.1	385	5.0
800	5.1	2.6	2.2	794	6.8	767	7.8
1600	7.6	5.2	5.4	1586	9.0	1537	10.8
3200	9.9	7.6	8.5	3170	11.5	3073	12.8
6400	12.3	10.0	11.5	6336	13.3	6145	12.0

This problem has also been solved in [24] using the eighth-order, eight-stage RKN (H8) method constructed by Hairer [15], the eleven-stage method of order 10 (H10) given in Hairer [15],the parallel-Iterated RKN (PIRKN III) method of order eight and eight order block third derivative BHTDA in [17]. We have chosen to compare these methods of order 8 with our method of order 9, because the orders of the methods are very close. The results obtained using the $H8$ PIRKN III methods are reproduced in table 11 and compared with the results given by BHI. It is seen from table 4 that the BHI performs better than those in [17] and in terms of accuracy (smaller errors) and efficiency (smaller NFEs).

6. Conclusion

Four step block hybrid integrator has been developed which is used together with additional methods in the block form (10) to simultaneously solve (1) directly without first adapting the second order IVP to an equivalent first order system. The integrators are implemented without the need for starting values or predictors. The efficiency of the BHI has been demonstrated on four numerical examples. Details of the numerical results are displayed in Tables 1-4.

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