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ON THE FUNDAMENTAL THEORY OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE

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Abstract: A special class of nonlinear non-autonomous systems ordinary differential equations with variable structure and impulses is studied in the paper. Right hand side of these equations are successively chosen by set f, consisting of infinitely many functions. There is $f = \{f_i = f_i(t,x),\ i = 1,2,...\}$. The main elements of any system such type is a set of linear switching functions: $\varphi = \{\varphi_i = \varphi_i(x),\ i = 1,2,...\}$ and a set of impulsive functions: $I = \{I_i = I_i(x),\ i = 1,2,...\}$. Each one of the switching and impulsive functions corresponds to the right side $f_i,\ i = 1,2,....$ The consecutive i-th change in the right side of this system (changing f_i with f_{i+1}) and the corresponding impulsive effect on the solution $x(t_i) \to x(t_i + 0) = x(t_i) + I_i\big(x(t_i)\big)$ are performed in the so-called i-th moment of switching $t_i,\ i = 1,2,....$ Just at this moment, the solution cancels φ_i, i.e. $\varphi_i\big(x(t_i)\big) = 0,\ i = 1,2,....$ The main research goal is to identify the reasons for which such impulsive systems have solutions which are not continuable up to infinity. The case of non continuability of the solutions (or as it is accepted to say death of the solutions) due to the impulsive effects is investigated in the paper.

AMS Subject Classification: 34A37
Key Words: impulsive systems, switching functions, “death of the solutions

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1. Introduction

The differential equations with impulses are introduced by V. Milman and A. Myshkis (see [15]). The qualities of the solutions of a special class of differential equations with variable structure and impulses are studied firstly in [14]. The differential equations, studied in this paper, are introduced in [4] and [5]. In article [5], for the same type equations, which are investigated in this paper, the relationship between the solution of the initial problem and external interferences in the switching functions is studied. Sufficient conditions are found under which "relatively small changes" of the switching functions cause "small controlled changes" in the solution. The results obtained of [5] are applied in [4] for studying the dynamics of hydraulic valve shutter. The change in the movement of the valve shutter, corresponding to the "small deformation" of its bed is evaluated.

The applications of the equations with impulsive effects are numerous. We will point: [2], [3], [7], [12], [13], [18], [19] and [20]. The equations with variable structure are applied mainly in theory of control: [1], [10], [11], [16] and [17]. In papers [6], [8] and [9] are given applications of differential equations with variable structure and impulses.

Let $\|.\|$ and $\langle ., . \rangle$ be the Euclidean norm and scalar product in R^n, respectively.

The object of investigation is the following initial problem:

$$\frac{dx}{dt} = f_i(t,x), \quad if\ \langle a_i, x(t) \rangle \neq \alpha_i,\ i.e.\ t_{i-1} < t < t_i; \tag{1}$$

$$\big\langle a_i, x(t_i) \big\rangle = \alpha_i,\ i = 1, 2, ...; \tag{2}$$

$$x(t_i + 0) = x(t_i) + I_i\big(x(t_i)\big); \tag{3}$$

$$x(t_0) = x_0, \tag{4}$$

where:

- The functions $f_i : R^+ \times D \to R^n$;
- The phase space D of system considered is nonempty set in R^n;
- The vectors $a_i = (a_i^1, a_i^2, ..., a_i^n) \in R^n$ and $a_i \neq 0$;
- The constants $\alpha_i \in R$;
- The functions $I_i : D \to R^n$;
- The functions $(Id + I_i) : D \to D,\ Id$ is the identity in R^n;

– The initial point $(t_0, x_0) \in R^+ \times D$ and $\langle a_1, x_0 \rangle \neq \alpha_1$.

The solution of initial problem is a piecewise continuous function with jump discontinuouty at $t_1, t_2, ...$ This solution is continuous on the left at any point in its domain. The points $t_1, t_2, ...$ are named moments of switching. The functions $I_1, I_2, ...$ are called impulsive. As can be seen from (1) and (2), the functions $\varphi_i(x) = \langle a_i, x \rangle - \alpha_i,\ i = 1, 2, ...$, are linear, and their corresponding sets:

$$\Phi_i = \{x \in D; \langle a_i, x \rangle = a_i^1 x^1 + a_i^2 x^2 + ... + a_i^n x^n = \alpha_i\},\ i = 1, 2, ...,$$

are parts of the hyperplanes in phase space. The functions $\varphi_1, \varphi_2, ...$ and sets $\Phi_1, \Phi_2, ...$, are called switching functions and switching sets.

The following notations are used:

– $f = \{f_1, f_2, ...\},\ \varphi = \{\varphi_1, \varphi_2, ...\},\ I = \{I_1, I_2, ...\}$;

– $x(t; t_0, x_0)$ is a solution of problem (1), (2), (3), (4);

– $x_i(t; t_0, x_0)$ is a solution of problem with fixed structure and without impulses

$$\frac{dx}{dt} = f_i(t, x), \quad x(t_0) = x_0, \quad i = 1, 2, ...; \tag{5}$$

– The curve $\gamma(t_0, x_0) = \{x(t; t_0, x_0);\ t \in J(t_0, x_0, f)\}$ is the trajectory of problem studied, where $J(t_0, x_0, f)$ is the maximum interval of existence of the solution;

– The curve $\gamma_i(t_0, x_0) = \{x_i(t; t_0, x_0);\ t \in J(t_0, x_0, f_i)\}$ is the trajectory of problem (5), where $J(t_0, x_0, f_i)$ is the maximum interval of existence of the solution, $i = 1, 2, ...$

Further, we will use the following conditions:

H1 The functions $f_i \in C[R^+ \times D, R^n],\ i = 1, 2, ...$

H2 The functions $I_i \in C[\Phi_i, R^n]$ and $(Id + I_i) : \Phi_i \to D,\ i = 1, 2, ...$

H3 For any point $(t_0, x_0) \in R^+ \times D$ and for each $i = 1, 2, ...$, the solution of initial problem (5) exists and is unique for $t \geq t_0$.

H4 The equalities $\|a_i\| = 1,\ i = 1, 2, ...$, are satisfied.

H5 The next inequalities are valid

$$\Big(\big\langle a_i,(Id+I_{i-1})(x)\big\rangle-\alpha_i\Big).\big\langle a_i,f_i(t,x)\big\rangle<0,\ (t,x)\in R^+\times D,\ i=1,2,...,$$

where $I_0(x)=0,\ x\in D$.

H6 There exist constants $C_{\langle a_i,f_i\rangle}>0$, such that

$$\big(\forall (t,x)\in R^+\times D\big)\Rightarrow\big\|\big\langle a_i,f_i(t,x)\big\rangle\big\|\geq C_{\langle a_i,f_i\rangle},\ i=1,2,....$$

H7 There exist constants $C_{a_i}>0$, such that

$$(\forall x\in\Phi_i)\Rightarrow\big|\big\langle a_{i+1},(Id+I_i)(x)\big\rangle-\alpha_i\big|\leq C_{a_{i+1}},\ i=1,2,....$$

H8 The series

$$\sum_{i=1}^{\infty}\frac{C_{a_{i+1}}}{C_{\langle a_{i+1},f_{i+1}\rangle}}$$

is convergent.

2. Main results

Theorem 1. *Let the conditions H1 - H6 be fulfilled.*

Then the trajectory of problem (1), (2), (3), (4) meets each one of the hyperplanes $\Phi_i,\ i=1,2,....$

Proof. We will show that the trajectory of the problem considered meets hyperplane Φ_1. From condition H5, it follows that one of the following two cases is satisfied:

Case 1. $\big(\langle a_1,x\rangle-\alpha_1\big)<0$ for $x\in D$ and $\big\langle a_1,f_1(t,x)\big\rangle>0$ for $(t,x)\in R^+\times D$;

Case 2. $\big(\langle a_1,x\rangle-\alpha_1\big)>0$ for $x\in D$ and $\big\langle a_1,f_1(t,x)\big\rangle<0$ for $(t,x)\in R^+\times D$.

Here, we look at the second case. The first case is considered similarly. We introduce a function $\phi_1(t)=\langle a_1,x_1(t;t_0,x_0\rangle-\alpha_1$, where $x_1(t;t_0,x_0)$ is a solution of problem (5) for $i=1$. The function is defined for $t\in J(t_0,x_0,f_1)=[t_0,\infty)$. We have

$$\phi_1(t_0)=\big\langle a_1,x_1(t_0;t_0,x_0)\big\rangle-\alpha_1=\langle a_1,x_0\rangle-\alpha_1>0.$$

According to condition H6, it is satisfied

$$\begin{aligned}\frac{d}{dt}\phi_1(t) &= \Big\langle a_1, \frac{d}{dt}x_1(t;t_0,x_0)\Big\rangle \\ &= \big\langle a_1, f_1\big(t, x_1(t;t_0,x_0)\big)\big\rangle \\ &= -\big|\big\langle a_1, f_1\big(t, x_1(t;t_0,x_0)\big)\big\rangle\big| \\ &\leq -C_{\langle a_1, f_1\rangle} = -const < 0.\end{aligned}$$

From the fact

$$\phi_1(t_0) > 0 \quad and \quad \frac{d}{dt}\phi_1(t) \leq -const < 0 \; for \; t > t_0,$$

it follows that there exists a point $t_1 > t_0$ such that

$$\big\langle a_1, x_1(t_1;t_0,x_0)\big\rangle - \alpha_1 = \phi_1(t_1) = 0.$$

This means that at the moment t_1, trajectory $\gamma_1(t_0,x_0)$ meets hyperplane Φ_1. Given that

$$\gamma(t_0,x_0) \equiv \gamma_1(t_0,x_0) \; for \; t_0 \leq t \leq t_1,$$

we conclude that the trajectory of problem (1), (2), (3), (4) also meets the hyperplane Φ_1 at moment t_1.

Assume that the trajectory of problem investigated consistently meets the hyperplanes $\Phi_1, \Phi_2, ..., \Phi_i$ at the moments $t_1, t_2, ..., t_i$, respectively. It is fulfilled $t_1 < t_2 < ... < t_i$. We will show that the trajectory $\gamma_{i+1}\big(t_0, x(t_i+0;t_0,x_0)\big)$ meets hyperplane Φ_{i+1}, from which it follows that the same is true for the studied trajectory $\gamma(t_0,x_0)$. Again, taking into account condition H5, without loss of generality, we will suppose that the following inequalities are valid:

$$\big\langle a_{i+1}, (Id+I_i)(x)\big\rangle - \alpha_{i+1} > 0 \text{ for } x \in D \; and \; \big\langle a_{i+1}, f_{i+1}(t,x)\big\rangle \; for \; (t,x) \in R^+ \times D.$$

We consider the function ϕ_{i+1}, defined by

$$\phi_{i+1}(t) = \big\langle a_{i+1}, x_{i+1}\big(t;t_i, x(t_i+0;t_0,x_0)\big)\big\rangle - \alpha_{i+1}, \quad t \geq t_i. \tag{6}$$

We have

$$\begin{aligned}\phi_{i+1}(t_i) &= \big\langle a_{i+1}, x_{i+1}\big(t_i;t_i, x(t_i+0;t_0,x_0)\big)\big\rangle - \alpha_{i+1} \\ &= \big\langle a_{i+1}, x(t_i+0;t_0,x_0)\big\rangle - \alpha_{i+1} \\ &= \big\langle a_{i+1}, x(t_i;t_0,x_0) + I_i\big(x(t_i;t_0,x_0)\big)\big\rangle - \alpha_{i+1}\end{aligned}$$

$$= \left\langle a_{i+1}, (Id + I_i)\big(x(t_i; t_0, x_0)\big)\right\rangle - \alpha_{i+1} > 0.$$

For $t > t_i$, it is satisfied

$$\begin{aligned} \frac{d}{dt}\phi_{i+1}(t) &= \left\langle a_{i+1}, f_{i+1}\big(t, x_{i+1}\big(t; t_i, x(t_i + 0; t_0, x_0)\big)\big)\right\rangle \\ &= -\left|\left\langle a_{i+1}, f_{i+1}\big(t, x_{i+1}\big(t; t_i, x(t_i + 0; t_0, x_0)\big)\big)\right\rangle\right| \\ &\leq -C_{\langle a_{i+1}, f_{i+1}\rangle} = -const < 0. \end{aligned}$$

Therefore, there exists a point $t_{i+1} > t_i$, such that

$$\phi_{i+1}(t_{i+1}) = 0 \;\Leftrightarrow\; \left\langle a_{i+1}, x_{i+1}\big(t_{i+1}; t_i, x\big(t_i + 0; t_0, x_0)\big)\right\rangle - \alpha_{i+1} = 0.$$

The last equality shows that trajectory $\gamma_{i+1}\big(t_i, x(t_i + 0; t_0, x_0)\big)$ meets hyperplane Φ_{i+1} at the moment t_{i+1}. The same applies to the trajectory $\gamma(t_0, x_0)$.

The proof of the theorem follows by induction.

The theorem is proved.

Theorem 2. *Let the conditions H1 - H7 be fulfilled.*
Then the next estimates are valid.

$$t_{i+1} - t_i \leq \frac{C_{a_{i+1}}}{C_{\langle a_{i+1}, f_{i+1}\rangle}}, \; i = 1, 2, \ldots.$$

Proof. Let i be an arbitrary natural number. We consider the function ϕ_{i+1}, defined by equality (7). Directly we obtain the next equality

$$\phi_{i+1}(t) = \begin{cases} \left\langle a_{i+1}, x(t_i + 0; t_0, x_0)\right\rangle - \alpha_{i+1} \\ \quad = \left\langle a_{i+1}, x(t_i; t_0, x_0) + I_i\big(x(t_i; t_0, x_0)\big)\right\rangle - \alpha_{i+1}, \quad t = t_i; \\ \left\langle a_{i+1}, x(t; t_0, x_0)\right\rangle - \alpha_{i+1}, \quad t_i < t \leq t_{i+1}. \end{cases}$$

Again, we suppose that inequalities (6) are valid. Using condition H7, we receive

$$\begin{aligned} \phi_{i+1}(t_{i+1}) &- \phi_{i+1}(t_i) \\ &= \left\langle a_{i+1}, x(t_{i+1}; t_0, x_0)\right\rangle - \left\langle a_{i+1}, x(t_i + 0; t_0, x_0)\right\rangle \\ &= -\left\langle a_{i+1}, x(t_i; t_0, x_0) + I_i\big(x(t_i; t_0, x_0)\big)\right\rangle + \alpha_{i+1} \\ &= \left|\left\langle a_{i+1}, (Id + I_i)\big(x(t_i; t_0, x_0)\big)\right\rangle - \alpha_{i+1}\right| \\ &\leq C_{a_{i+1}}. \end{aligned} \tag{7}$$

On the other hand, using the conditions H6 and H4, consistently we obtain

$$\phi_{i+1}(t_{i+1}) \quad - \quad \phi_{i+1}(t_i)$$

$$
\begin{aligned}
&= \frac{d}{dt}\phi_{i+1}(t^*)(t_{i+1}-t_i)\\
&= \frac{d}{dt}\big(\langle a_{i+1}, x(t^*;t_0,x_0)\rangle - \alpha_{i+1}\big).(t_{i+1}-t_i)\\
&= \frac{d}{dt}\big(\langle a_{i+1}, x_{i+1}\big(t^*;t_0,x(t_i+0;t_0,x_0)\big)\rangle - \alpha_{i+1}\big).(t_{i+1}-t_i)(8)\\
&= \big\langle a_{i+1}, f_{i+1}\big(t^*, x_{i+1}\big(t^*;t_0,x(t_i+0;t_0,x_0)\big)\big)\big\rangle.(t_{i+1}-t_i)\\
&\geq \|a_{i+1}\|.C_{\langle a_{i+1},f_{i+1}\rangle}.(t_{i+1}-t_i)\\
&= C_{\langle a_{i+1},f_{i+1}\rangle}.(t_{i+1}-t_i).
\end{aligned}
$$

where point t^* satisfies the inequalities $t_i < t^* < t_{i+1}$. By (8) and (9), it follows the wanted estimate.

The theorem is proved.

Theorem 3. *Let the conditions H1 - H8 be fulfilled.*
Then the solutions of system (1), (2), (3) die due to the impulsive effects.

Proof. It is valid

$$J(t_0,x_0,f) = [t_0,t_1] \cup (t_1,t_2] \cup (t_2,t_3] \cup ... = [t_0,t^0),$$

where

$$
\begin{aligned}
t^0 &= \lim_{i\to\infty} t_i\\
&= t_1 + \lim_{i\to\infty}\big((t_2-t_1)+(t_3-t_2)+...+(t_i-t_{i-1})\big)\\
&= t_1 + \sum_{i=1}^{\infty}(t_{i+1}-t_i)\\
&\leq t_1 + \sum_{i=1}^{\infty}\frac{C_{a_{i+1}}}{C_{\langle a_{i+1},f_{i+1}\rangle}} < \infty.
\end{aligned}
$$

The theorem is proved.

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