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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A QUASILINEAR ELLIPTIC EQUATION INVOLVING p-LAPLACIAN

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Abstract: In the present paper, using direct variational approach, the existence and uniqueness of solutions for a quasilinear elliptic equation involving the p-Laplacian is obtained.

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1. Introduction

We study the existence and uniqueness of solutions to the following Dirichlet boundary problem

$$\begin{cases} -\Delta_p u = f(u) + h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{(P)}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $2 \leq p < N$, $f : \mathbb{R} \to \mathbb{R}$ is

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a continuous function and $h \in L^{p/p-1}(\Omega)$.

The importance of problem (**P**) arises from the existence of the p-Laplacian or the p-Laplace operator $\triangle_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. Obviously, when $p = 2$, $\triangle_2 = \triangle$ is the usual Laplace operator. However, in case $p \neq 2$ the situation is very crucial, as for example, one encounters the lack of the Hilbert structure of the space $W_0^{1,p}(\Omega)$ when passing from $p = 2$ to $p \neq 2$.

It is well known that problems involving the p-Laplacian appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, nonlinear elasticity, and reaction diffusions. For discussions about problems modelled by these boundary value problems, see, for example, [4]. One of the most widely used results for solving problem (**P**) is the mountain pass theorem. In order to apply this theorem, it is necessary that the Euler-Lagrange functional associated to the problem has the Palais-Smale property. One way to ensure this is to assume that f satisfies some Ambrosetti-Rabinowitz-type condition (see, e.g., [2, 5]). Another technique used for obtaining solutions of problem (**P**) is the blowup method due to Gidas and Spruck [6]. In order to use any of the techniques above, it is necessary that the nonlinearity f has subcritical growth. The object of this paper is to study problem (**P**) for nonlinearities f which do not necessarily satisfy the classical conditions, such as Ambrosetti-Rabinowitz condition, but are limited by functions that do satisfy some specific conditions.

$L^p(\Omega)$, for $1 \leq p < \infty$, is the Lebesgue space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u|^p \, dx < +\infty$, while $L^\infty(\Omega)$ is the space of measurable functions such that $\operatorname{ess\,sup}_{x\in\Omega} |u(x)| < +\infty$. The norms that make $L^p(\Omega)$ Banach spaces are, respectively,

$$|u|_p = \left(\int_\Omega |u|^p \, dx\right)^{1/p} \quad \text{and} \quad |u|_\infty = \operatorname{ess\,sup} |u(x)| .$$

Let $W^{1,p}(\Omega)$ be the usual Sobolev space, i.e.,

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$$

which is endowed with norm

$$\|u\| = \left(\int_\Omega (|\nabla u|^p + |u|^p) \, dx\right)^{1/p} . \tag{1.1}$$

Then $W^{1,p}(\Omega)$ is a Banach space. $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

In $W_0^{1,p}(\Omega)$ we use the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}. \tag{1.2}$$

Thanks to the Poincaré inequality, it is not difficult to see that (1.1) and (1.2) are equivalent. Therefore, in the sequel the norm in $W_0^{1,p}(\Omega)$ will be denoted by $\|\cdot\|$.

Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Then $W_0^{1,p}(\Omega)$ is embedded continuously in $L^q(\Omega)$, denoted by $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for every $q \in [1, p^*]$, where $p^* = Np/N - p$. The embedding is compact if and only if $q \in [1, p^*)$ (see [1]).

$u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathbf{P})$ if any $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f(u) \varphi dx + \int_\Omega h \varphi dx.$$

The energy functional corresponding to problem $(\mathbf{P})$ is defined as $J : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(u) \, dx - \int_\Omega hu dx,$$

where $F(t) = \int_0^t f(s) \, ds$. It is well known that weak solutions of $(\mathbf{P})$ correspond to critical points of the functional J.

The main result of the present paper is:

Theorem 1. *Assume that the following conditions hold:*

(f_0) $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function, and assume that there exists* $a, b > 0$ *such that*

$$|f(t)| \leq a + b |t|^{q-1}$$

for all $t \in \mathbb{R}$*, where* $1 < q < p^*$.

(f_1) $f(t) t \leq 0$ *and* $(f(t) - f(s))(t - s) \leq 0$ *for all* $t, s \in \mathbb{R}$.

Then problem $(\mathbf{P})$ *has exactly one solution.*

Remark 1. As an example of function f satisfying the assumptions of Theorem 1, one can take $f(t) = -|t|^{r-2} t$ with $r \in (1, p^*)$.

2. Main Results

First, we give the following well-known Propositions which are necessary through the present paper (see, e.g., [3, 7, 9]).

Proposition 1. *Let X be a Banach space and let $I : X \to \mathbb{R}$ be a differentiable functional. Assume that for all $u, v \in X$,*

$$\left\langle I'(u) - I'(v), u - v \right\rangle \geq 0.$$

Then I is convex. If the strict inequality holds when $u \neq v$, then I is strictly convex.

Proposition 2. *Let functional $I : X \to \mathbb{R}$ be strictly convex and differentiable. Then I has at most one critical point in X.*

To obtain the result of Theorem 1, we need to show that the following two lemmas hold.

Lemma 1. (i) *The functional J is well-defined on $W_0^{1,p}(\Omega)$.*
(ii) *The functional J is of class $C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$ and*

$$\left\langle J'(u), \varphi \right\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx - \int_\Omega f(u) \varphi dx - \int_\Omega h \varphi dx,$$

for all $u, \varphi \in W_0^{1,p}(\Omega)$.

Proof. (i) From (f_0), (f_1), continuous embeddings and the Hölder inequality we have

$$\begin{aligned} J(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega F(u) dx - \int_\Omega h u dx \\ &\leq \frac{1}{p} \|u\|^p + a \int_\Omega |u| dx + \frac{b}{q} \int_\Omega |u|^q dx - |h|_{\frac{p}{p-1}} |u|_p \\ &\leq \frac{1}{p} \|u\|^p + c_1 \|u\|^q + c_2 \|u\|, \end{aligned}$$

this implies that J is well-defined on $W_0^{1,p}(\Omega)$.

(ii) Let define

$$J(u) = \int_\Omega \left(\frac{1}{p} |\nabla u|^p - F(u) - hu \right) dx := \int_\Omega G(u) dx.$$

First, we will show that J is Gâteaux differentiable. Therefore we have to prove that for fixed $u, v \in W_0^{1,p}(\Omega)$,

$$\lim_{t\to 0}\int_\Omega \frac{G(u+tv)-G(u)}{t} = \int_\Omega \Big(|\nabla u|^{p-2}\nabla u\cdot\nabla v-(f(u)+h)v\Big)\,dx.$$

After elementary calculations, one can easily show that, for almost every $x\in\Omega$,

$$\begin{aligned}&\lim_{t\to 0}\frac{G(u(x)+tv(x))-G(u(x))}{t}\\ =\ & |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)-(f(u(x))+h(x))v(x).\end{aligned}$$

Then, by the Lagrange theorem, there exists a real number θ such that $|\theta|\le|t|$ and

$$\begin{aligned}&\left|\frac{G(u(x)+tv(x))-G(u(x))}{t}\right|\\ =&\Big||\nabla(u(x)+\theta v(x))|^{p-2}\nabla(u(x)+\theta v(x))\cdot\nabla v(x)\\ &-(f(u(x)+\theta v(x))+h(x))v(x)\Big|\\ \le&|\nabla(u(x)+\theta v(x))|^{p-1}|\nabla v(x)|+\Big(a+b|u(x)+\theta v(x)|^{q-1}\Big)|v(x)|+|h(x)v(x)|\\ \le&|\nabla(u(x)+v(x))|^{p-1}|\nabla v(x)|+\Big(a+b|u(x)+v(x)|^{q-1}\Big)|v(x)|+|h(x)||v(x)|\\ \le&|\nabla u(x)|^{p-1}|\nabla v(x)|+|\nabla v(x)|^p+|u(x)|^{q-1}|v(x)| \qquad (2.1)\\ &+|v(x)|^q+|v(x)|+|h(x)||v(x)|.\end{aligned}$$

By the Hölder inequality we get

$$\int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx\le\Big||\nabla u|^{p-1}\Big|_{\frac{p}{p-1}}|\nabla v|_p,$$

and

$$\int_\Omega |u|^{q-1}|v|\,dx\le\Big||u|^{q-1}\Big|_{\frac{q}{q-1}}|v|_q,\quad \int_\Omega |h||v|\,dx\le|h|_{\frac{p}{p-1}}|v|_p.$$

From the above inequalities, one concludes that the expression (2.1) is in $L^1(\Omega)$. Therefore by the dominated convergence theorem we have

$$\lim_{t\to 0}\int_\Omega \frac{G(u+tv)-G(u)}{t} = \int_\Omega \Big(|\nabla u|^{p-2}\nabla u\cdot\nabla v-(f(u)+h)v\Big)\,dx.$$

Since the right-hand side, as a function of v, is continuous linear functional on $W_0^{1,p}(\Omega)$, it is the Gâteaux differential of J.

Now, we will prove that $J' : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)^*$ is continuous. Assume $u_k \to u$ in $W_0^{1,p}(\Omega)$. Up to a subsequence, we may assume that

$u_k \to u$ in $L^p(\Omega)$ and in $L^q(\Omega)$ as $k \to \infty$;

$u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$.

Then, by the Hölder inequality we have

$$\begin{aligned}
&\left|\left\langle J'(u_k) - J'(u), v\right\rangle\right| \\
\leq\ & \int_\Omega \left|\left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla v + (f(u) - f(u_k))\,v\right| dx \\
\leq\ & \int_\Omega \left||\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right| |\nabla v|\,dx + \int_\Omega |f(u_k) - f(u)|\,|v|\,dx \\
\leq\ & \int_\Omega \left(|\nabla u_k|^{p-1} + |\nabla u|^{p-1}\right)|\nabla v|\,dx + \int_\Omega |f(u_k) - f(u)|\,|v|\,dx.
\end{aligned}$$

Since $u_k \to u$ in $L^p(\Omega)$, it is obvious that $\left(|\nabla u_k|^{p-1} + |\nabla u|^{p-1}\right)|\nabla v| \in L^1(\Omega)$. Let's proceed for the second term. By the Hölder inequality we obtain

$$\int_\Omega |f(u_k) - f(u)|\,|v|\,dx \leq \left(\int_\Omega |f(u_k) - f(u)|^{q/q-1}\,dx\right)^{q-1/q}\left(\int_\Omega |v|^q\,dx\right)^{1/q}.$$

From (f_0) we have

$$|f(u_k) - f(u)|^{q/q-1} \leq c\left(1 + |u_k|^{q-1} + |u|^{q-1}\right)^{q/q-1}.$$

Since $u_k \to u$ in $L^q(\Omega)$, there exists $\varphi \in L^q(\Omega)$ such that $|u_k(x)| \leq \varphi(x)$ a.e. in Ω and for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned}
|f(u_k) - f(u)|^{q/q-1} &\leq c_1\left(1 + |\varphi|^{q-1} + |u|^{q-1}\right)^{q/q-1} \\
&\leq c_2\left(1 + |\varphi|^q + |u|^q\right) \in L^1(\Omega).
\end{aligned}$$

On the other hand, taking into account that $u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$, we also get

$$\lim_{k\to\infty}\left||\nabla u_k(x)|^{p-2}\nabla u_k(x) - |\nabla u(x)|^{p-2}\nabla u(x) - (f(u_k(x)) - f(u(x)))\right| = 0.$$

If we consider the above inequalities and apply the dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\int_\Omega \left||\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u - (f(u_k) - f(u))\right| = 0,$$

which implies

$$\lim_{k\to\infty} \sup \left\| J'(u_k) - J'(u) \right\| = 0.$$

So we deduce that Gâteaux differential of J is continuous, i.e. J is of class $C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$. □

Lemma 2. *(i) The functional J is coercive.*
(ii) The functional J is strictly convex.

Proof. (i) From (f_1), it is clear that $F(t) \leq 0$ for all $t \in \mathbb{R}$, and hence we have

$$\begin{aligned} J(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega F(u)\, dx - \int_\Omega hudx \\ &\geq \frac{1}{p}\|u\|^p - c\|u\|, \end{aligned}$$

this implies that J is coercive.

(ii) For all $u, v \in W_0^{1,p}(\Omega)$ for $u \neq v$, from (f_1) we have

$$\begin{aligned} &\left\langle I'(u) - I'(v), u - v \right\rangle \\ = &\int_\Omega \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot (\nabla u - \nabla v)\, dx \\ &- \int_\Omega (f(u) - f(v))(u - v)\, dx \\ \geq &\int_\Omega \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot (\nabla u - \nabla v)\, dx. \end{aligned}$$

Next, we apply the following well-known inequality (see [8]), for any $\xi, \eta \in \mathbb{R}^N$

$$\left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta \right) \cdot (\xi - \eta) \geq 2^{-r} |\xi - \eta|^r, \; r \geq 2.$$

Therefore, one easily concludes that

$$\left\langle I'(u) - I'(v), u - v \right\rangle \geq 2^{-p} \int_\Omega |\nabla(u - v)|^p \, dx = 2^{-p} \|u - v\|^p > 0.$$

This shows that J is strictly convex. □

Proof of Theorem 1. As the functional J is continuous, convex and coercive, it has a global minimum point, which is a critical point. Furthermore, since J is strictly convex and differentiable, by Proposition 4, J must have only one critical point. □

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York (1975).

[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *Journal of Functional Analysis*, **14**, No. 4 (1973), 349-38.

[3] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners Existence Results via the Variational Approach*, Springer-Verlag London Limited, London (2011).

[4] J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Volume I. Elliptic Equations, vol. 106 of Research Notes in Mathematics, Pitman, Boston, Mass, USA (1985).

[5] D.G. de Figueiredo, J.-P. Gossez, P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, *Journal of Functional Analysis*, **199**, No. 2 (2003), 452-467.

[6] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Communications on Pure and Applied Mathematics*, **34**, No. 4 (1981), 525-598.

[7] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, River Edge (2003).

[8] S. Kichenassamy, L. Veron, Singular solutions of the p-Laplace equation, *Math. Ann.*, **275** (1985), 599-615.

[9] M. Willem, *Minimax Theorems*, Birkhauser, Basel (1996).