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INTERVAL OSCILLATION THEORY FOR LINEAR MATRIX HAMILTONIAN SYSTEMS

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Abstract: By means of Riccati transformation technique and averaging method, interval oscillation theorems are established for linear matrix self-adjoint Hamiltonian system

$$\begin{cases} U'(x) = A(x)U(x) + B(x)V(x), \\ V'(x) = C(x)U(x) - A^*(x)V(x), \end{cases}$$

that are different from most known ones in the sense that properties of A, B and C are only needed on a sequence of subintervals of $[x_0, \infty)$, rather than on the whole half-line. Examples that dwell upon the importance of our results are also included.

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1. Introduction

Recently, interval oscillation theory for ordinary differential equations has paid much attention by several authors [2, 7, 8, 9, 10]. But very few work has been done to linear Hamiltonian matrix systems. Motivated by this, in this paper, by using Riccati techniques and the averaging methods, we establish interval oscillation theory (Theorems 2.1-2.4) for the system

$$\begin{cases} U'(x) = A(x)U(x) + B(x)V(x), \\ V'(x) = C(x)U(x) - A^*(x)V(x), \end{cases} \tag{1.1}$$

where $A(x), B(x) = B^*(x) > 0$ and $C(x) = C^*(x)$ are $n \times n$ real continuous matrix functions on the interval $I = [x_0, \infty)$, which include as special cases

$$\big(P(x)U'(x)\big)' + Q(x)U(x) = 0, \tag{1.2}$$

and

$$\big(p(x)y'(x)\big)' + q(x)y(x) = 0. \tag{1.3}$$

Our results extend, improve and genralize a number of existing results [5, 6, 11, 12, 13, 14, 15]. Some of these criteria are given by the behavior of (1.1) (that is, A, B and C) only on a sequence of subintervals of $[x_0, \infty)$ and can be applied to extreme cases such as $\lim_{x\to\infty} \lambda_1(-\int_{x_0}^x C(t)dt) = \infty$ or $\int_{x_0}^\infty \mathrm{tr}C(t)dt = \infty$. Finally, several examples that illustrate the importance of our results are included.

Before entering into the discussion of our results, we now recall some standard definitions from the earlier literature.

Definition 1.1. A solution $(U(x), V(x))$ of system (1.1) is said to be *nontrivial* if $\det U(x) \neq 0$ for at least one $x \in [x_0, \infty)$.

Definition 1.2. A nontrivial solution $(U(x), V(x))$ of system (1.1) is said to be *prepared* or *self-adjoint* if

$$U^*(x)V(x) - V^*(x)U(x) = 0, \quad x \geq x_0,$$

where A^* denotes the transpose of A.

Definition 1.3. System (1.1) is *oscillatory* on $[x_0, \infty)$ if there is a nontrivial prepared solution $(U(x), V(x))$ of system (1.1) which has the property that $\det U(x)$ vanishes on $[T, \infty)$ for each $T > x_0$. Otherwise, (1.1) is said to be *nonoscillatory*.

We remark that if system (1.1) is oscillatory then every nontrivial prepared solution $(\overline{U}(x), \overline{V}(x))$ of system (1.1) has the property that $\det U(x)$ vanishes on $[T, \infty)$ for each $T > x_0$ (see [4, Theorem 8.1, p. 303]).

Hereafter we denote the trace of an $n \times n$ matrix A by tr$A$. Further, we denote $n \times n$ identity matrix by E_n and the eigenvalues of an $n \times n$ Hermitian or symmetric matrix A (in the increasing order) by

$$\lambda_{\min}[A] = \lambda_n[A] \le ... \le \lambda_1[A] = \lambda_{\max}[A].$$

We now state a lemma (see [3]) which is needed for proving the results in following sections.

Lemma 1.1. *Let P and Q be Hermitian and let the eigenvalues $\lambda_i[P]$, $\lambda_i[Q]$ and $\lambda_i[P+Q]$ be arranged in increasing order as stated above. For each $i = 1, 2, ..., n$, we have*

$$\lambda_i[P] + \lambda_n[Q] \le \lambda_i[P+Q] \le \lambda_i[P] + \lambda_1[Q].$$

2. Interval Oscillation Theory

In the sequel we say that a function $H = H(x, s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D, R_+)$, where $D = \{(x, s) : -\infty < s \le x < \infty\}$, which satisfies

$$H(x, x) = 0, \quad H(x, s) > 0, \text{ for } x > s, \tag{2.1}$$

and has partial derivatives $\partial H/\partial x$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial x} = h_1(x, s) H(x, s)^{1/2} \text{ and } \frac{\partial H}{\partial s} = -h_2(x, s) H(x, s)^{1/2}, \tag{2.2}$$

where $h_1, h_2 \in C(D, R)$.

The following lemmas will be useful for establishing interval oscillation criteria for the system (1.1).

Lemma 2.1. *Suppose that $(U(x), V(x))$ is a solution of the system (1.1) such that $\det U(x) \neq 0$ on $[c, b)$. For any $\varphi \in C^1[x_0, \infty)$, let*

$$W(x) = v(x) \left[V(x) U^{-1}(x) + \varphi(x) E_n \right] \tag{2.3}$$

on $[c, b)$. Then for any $H \in X$,

$$\frac{1}{H(b,c)}\lambda_1\left[\int_c^b \{H(b,s)v(s)K(s)\right.$$
$$\left.-\frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)\right\} ds\Bigg]$$
$$\leq \lambda_1\left[W(c)\right], \quad (2.4)$$

where

$$v(x) = \exp\left(-2\int_{x_0}^x \varphi(s)ds\right),$$
$$T(x) = \left(A^* + B^{-1}AB + 2\varphi E_n - 2\varphi B\right)(x), \quad (2.5)$$

and

$$K(x) = \frac{1}{4}\Bigg((A^*)^2 B^{-1} + B^{-1}A^2 + B^{-1}ABA^*B^{-1} + A^*B^{-1}A + 4\varphi^2 B - 4\varphi' E_n$$
$$- 4C - 2\varphi\left[A + A^* + B^{-1}AB + BA^*B^{-1}\right]\Bigg)(x). \quad (2.6)$$

Proof. By differentiating the Hermitian matrix (2.3) and making use of (1.1), we obtain that for each $x \in [c,b)$, $W(x)$ satisfies the Riccati equation

$$W'(x)$$
$$= v'(x)\left[V(x)U^{-1}(x) + \varphi(x)E_n\right] + v(x)\left[V'U^{-1} - VU^{-1}U'U^{-1} + \varphi' E_n\right](x)$$
$$= -2\varphi W(x) + v(x)\left[(CU - A^*V)U^{-1} - VU^{-1}(AU + BV)U^{-1} + \varphi' E_n\right](x)$$
$$= -2\varphi W(x) + v(x)\left[C - A^*VU^{-1} - VU^{-1}A - VU^{-1}BVU^{-1} + \varphi' E_n\right](x)$$
$$= -\left[A^*W + WA + \frac{1}{v}WBW - \varphi(WB + BW - 2W)\right](x)$$
$$+ v(x)\left[C + \varphi(A + A^*) - \varphi^2 B + \varphi' E_n\right]. \quad (2.7)$$

If we replace x with s in the above equation and multiply the resulting equation by $H(x,s)$ and then integrating with respect to s from c to $x(> c)$ and rearranging, we obtain for $x \in (c,b)$,

$$-\int_c^x H(x,s)v(s)\left[C+\varphi\left(A+A^*\right)-\varphi^2 B+\varphi' E_n\right](s)ds$$

$$= -\int_c^x H(x,s)W'(s)ds$$

$$-\int_c^x H(x,s)\left[A^*W+WA+\frac{1}{v}WBW-\varphi\left(WB+BW-2W\right)\right](s)ds$$

$$= H(x,c)W(c)-\int_c^x h_2(x,s)H^{1/2}(x,s)W(s)ds$$

$$-\int_c^x H(x,s)\left[A^*W+WA+\frac{1}{v}WBW-\varphi\left(WB+BW-2W\right)\right](s)ds. \tag{2.8}$$

Set $R(x)=B^{1/2}(x)$,

$$\begin{aligned} G_1(x,s) &= h_2(x,s)H^{1/2}(x,s)\left(RWR\right)(s) \\ &\quad +H(x,s)\left(R\left[A^*W+WA-\varphi\left(WB+BW-2W\right)\right]R\right)(s) \\ &\quad +\frac{1}{v}H(x,s)\left(\left[RWR\right]\left[RWR\right]\right)(s), \end{aligned}$$

and

$$Q_1(x,s)$$
$$= H^{1/2}(x,s)\left[\frac{1}{\sqrt{v}}RWR+\sqrt{v}R^{-1}AR-\varphi\sqrt{v}R^{-1}BR+\varphi\sqrt{v}E_n\right](s)$$
$$+\frac{\sqrt{v(s)}}{2}h_2(x,s)E_n$$

for $c \le s \le x < b$. Then

$$
\begin{aligned}
&\left(Q_1^* Q_1\right)(x,s) \\
&= H(x,s)\left[\frac{1}{v}(RWR)(RWR) + RWAR - \varphi RWBR + \varphi RWR\right](s) \\
&+ H(x,s)\left[RA^*WR + vRA^*B^{-1}AR - \varphi vRA^*R + \varphi vRA^*R^{-1}\right](s) \\
&+ H(x,s)\left[-\varphi RBWR - \varphi vRAR + \varphi^2 vRBR - \varphi^2 vRBR^{-1}\right](s) \\
&+ H(x,s)\left[\varphi RWR + \varphi vR^{-1}AR - \varphi^2 vR^{-1}BR + \varphi^2 vE_n\right](s) \\
&+ h_2(x,s)H^{1/2}(x,s)(RWR)(s) + \frac{v(s)}{4}h_2^2(x,s)E_n \\
&+ \frac{v(s)}{2}\left(h_2 H^{1/2}\right)(x,s) \\
&\times \left[R^{-1}AR + RA^*R^{-1} - \varphi R^{-1}BR - \varphi RBR^{-1} + 2\varphi E_n\right](s)\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& -h_2(x,s)H^{1/2}(x,s)W(s) - \frac{1}{v(s)}H(x,s)(WBW)(s) \\
& -H(x,s)\left[A^*W + WA - \varphi(WB + BW - 2W)\right](s) \\
= \quad & -R^{-1}(s)G_1(x,s)R^{-1}(s) \\
= \quad & -R^{-1}(s)\left(Q_1^* Q_1\right)(x,s)R^{-1}(s) + \frac{v(s)}{4}h_2^2(x,s)B^{-1}(s) \\
& +\frac{v(s)}{2}\left(h_2 H^{1/2}\right)(x,s)\left[A^* + B^{-1}AB + 2\varphi E_n - 2\varphi B\right](s)B^{-1}(s) \\
& +v(s)H(x,s)\left[\varphi\left(A^*B^{-1} + B^{-1}A - A - A^*\right)\right. \\
& \left. +\varphi^2\left(B + B^{-1} - 2E_n\right) + A^*B^{-1}A\right](s) \\
= \quad & -R^{-1}(s)\left(Q_1^* Q_1\right)(x,s)R^{-1}(s) \\
& +\frac{v(s)}{4}\left[h_2(x,s) + H^{1/2}(x,s)\left[A^* + B^{-1}AB + 2\varphi E_n - 2\varphi B\right](s)\right]^2 B^{-1}(s)
\end{aligned}
$$

$$-\frac{v(s)}{4}H(x,s)\Bigg[2\varphi\left(A+A^*-B^{-1}AB-BA^*B^{-1}\right)$$

$$+\left(A^*\right)^2B^{-1}+B^{-1}A^2+B^{-1}ABA^*B^{-1}+A^*B^{-1}A\Bigg](s). \tag{2.9}$$

From (2.8) and (2.9) we have

$$\begin{aligned}
& -\int_c^x H(x,s)v(s)\left[C+\varphi\left(A+A^*\right)-\varphi^2B+\varphi'E_n\right](s)ds \\
= \quad & H(x,c)W(c)-\int_c^x R^{-1}(s)G(x,s)R^{-1}(s)ds \\
= \quad & H(x,c)W(c)-\int_c^x R^{-1}(s)\left(Q^*Q\right)(x,s)\,R^{-1}(s)ds \\
& +\int_c^x \frac{v(s)}{4}\left[h_2(x,s)+H^{1/2}(x,s)\left[A^*+B^{-1}AB+2\varphi E_n-2\varphi B\right](s)\right]^2 \\
& \times B^{-1}(s)ds \\
& -\int_c^x \frac{v(s)}{4}H(x,s)\Bigg[2\varphi\left(A+A^*-B^{-1}AB-BA^*B^{-1}\right) \\
& +\left(A^*\right)^2B^{-1}+B^{-1}A^2+B^{-1}ABA^*B^{-1}+A^*B^{-1}A\Bigg](s)ds. \qquad (2.10)
\end{aligned}$$

Set

$$S_1(x,c)=\int_c^x\Big\{H(x,s)v(s)K(s) \\ -\frac{v(s)}{4}\left[h_2(x,s)E_n+H^{1/2}(x,s)T\left(s\right)\right]^2B^{-1}(s)\Big\}\,ds,$$

where $K(x)$ and $T(x)$ are defined by (2.5) and (2.6), respectively, then we have

$$\begin{aligned}
& \int_c^x H(x,s)v(s)K(s)ds \\
= \quad & H(x,c)W(c)-\int_c^x R^{-1}(s)G_1(x,s)R^{-1}(s)ds
\end{aligned}$$

$$+\int_c^x \frac{v(s)}{4}\left[h_2(x,s)E_n + H^{1/2}(x,s)T(s)\right]^2 B^{-1}(s)ds, \quad (2.11)$$

and so

$$S_1(x,c) = H(x,c)W(c) - \int_c^x R^{-1}(s)G_1(x,s)R^{-1}(s)ds,$$

which yields

$$S_1(x,c) \le H(x,c)W(c). \quad (2.12)$$

Therefore

$$\lambda_1\left[S_1(x,c)\right] \le \lambda_1\left[H(x,c)W(c)\right] = H(x,c)\lambda_1\left[W(c)\right]. \quad (2.13)$$

Let $x \to b^-$ in (2.13) and dividing both sides by $H(b,c)$, then we have

$$\frac{1}{H(b,c)}\lambda_1\left[S_1(b,c)\right] \le \lambda_1\left[W(c)\right].$$

The proof is complete. □

Lemma 2.2. Suppose that $(U(x),V(x))$ is a solution of the system (1.1) such that $\det U(x) \neq 0$ on $(a,c]$. For any $\varphi \in C^1[x_0,\infty)$, let $W(t)$ be defined by (2.3) on $(a,c]$. Then for any $H \in X$,

$$\frac{1}{H(c,a)}\lambda_n\left[\int_a^c \left\{H(s,a)v(s)K(s) - \frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)\right\}ds\right] \le -\lambda_1\left[W(c)\right], \quad (2.14)$$

where $v(t), K(x)$ and $T(x)$ are defined as in Lemma 2.1.

Proof. Similar to the proof of Lemma 2.1, we multiply (2.7) by $H(s,x)$, integrate it with respect to s from $x(> a)$ to c, and use (2.1) and (2.2), then we get that

$$-\int_x^c H(s,x)v(s)\left[C + \varphi\left(A + A^*\right) - \varphi^2 B + \varphi' E_n\right](s)ds$$

$$= -\int_x^c H(s,x)W'(s)ds$$

$$
\begin{aligned}
& -\int_x^c H(s,x)\left[A^*W + WA + \frac{1}{v}WBW\right.\\
& \left.-\varphi\left(WB + BW - 2W\right)\right](s)ds \\
= \quad & -H(c,x)W(c) + \int_x^c h_1(x,s)H^{1/2}(x,s)W(s)ds \\
& -\int_x^c H(s,x)\left[A^*W + WA + \frac{1}{v}WBW\right.\\
& \left.-\varphi\left(WB + BW - 2W\right)\right](s)ds. \qquad (2.15)
\end{aligned}
$$

Set $R(x) = B^{1/2}(x)$,

$$
\begin{aligned}
G_2(s,x) \quad = \quad & -h_1(x,s)H^{1/2}(s,x)\left(RWR\right)(s) \\
& +H(s,x)\left(R\left[A^*W + WA - \varphi\left(WB + BW - 2W\right)\right]R\right)(s) \\
& +\frac{1}{v}H(s,x)\left(\left[RWR\right]\left[RWR\right]\right)(s),
\end{aligned}
$$

and

$$
\begin{aligned}
& Q_2(s,x) \\
& = H^{1/2}(s,x)\left[\frac{1}{\sqrt{v}}RWR + \sqrt{v}R^{-1}AR - \varphi\sqrt{v}R^{-1}BR + \varphi\sqrt{v}E_n\right](s) \\
& - \frac{\sqrt{v(s)}}{2}h_1(s,x)E_n
\end{aligned}
$$

for $a < x \leq s \leq b$. Then

$$
\begin{aligned}
& \left(Q_2^*Q_2\right)(x,s) \\
& = H(s,x)\left[\frac{1}{v}\left(RWR\right)\left(RWR\right) + RWAR - \varphi RWBR + \varphi RWR\right](s) \\
& + H(s,x)\left[RA^*WR + vRA^*B^{-1}AR - \varphi vRA^*R + \varphi vRA^*R^{-1}\right](s) \\
& + H(s,x)\left[-\varphi RBWR - \varphi vRAR + \varphi^2 vRBR - \varphi^2 vRBR^{-1}\right](s) \\
& + H(s,x)\left[\varphi vRWR + \varphi vR^{-1}AR - \varphi^2 vR^{-1}BR + \varphi^2 vE_n\right](s) \\
& - h_1(s,x)H^{1/2}(s,x)\left(RWR\right)(s) + \frac{v(s)}{4}h_1^2(s,x)E_n
\end{aligned}
$$

$$-\frac{v(s)}{2}\left(h_1 H^{1/2}\right)(s,x)$$
$$\times\left[R^{-1}AR+RA^*R^{-1}-\varphi R^{-1}BR-\varphi RBR^{-1}+2\varphi E_n\right](s)\ .$$

Hence,

$$\begin{aligned}
& h_1(s,x)H^{1/2}(s,x)W(s)-\frac{1}{v(s)}H(s,x)\left(WBW\right)(s) \\
& -H(s,x)\left[A^*W+WA-\varphi\left(WB+BW-2W\right)\right](s) \\
= \ & -R^{-1}(s)G_2(s,x)R^{-1}(s) \\
= \ & -R^{-1}(s)\left(Q_2^*Q_2\right)(s,x)R^{-1}(s)+\frac{v(s)}{4}h_1^2(s,x)B^{-1}(s) \\
& -\frac{v(s)}{2}\left(h_2H^{1/2}\right)(s,x)\left[A^*+B^{-1}AB+2\varphi E_n-2\varphi B\right](s)B^{-1}(s) \\
& +v(s)H(s,x)\left[\varphi\left(A^*B^{-1}+B^{-1}A-A-A^*\right)+\varphi^2\left(B+B^{-1}-2E_n\right)\right. \\
& \left.+A^*B^{-1}A\right](s) \\
= \ & -R^{-1}(s)\left(Q_2^*Q_2\right)(x,s)R^{-1}(s) \\
& +\frac{v(s)}{4}\left[h_1(s,x)\right. \\
& \left.-H^{1/2}(s,x)\left[A^*+B^{-1}AB+2\varphi E_n-2\varphi B\right](s)\right]^2B^{-1}(s) \\
& -\frac{v(s)}{4}H(x,s)\left[2\varphi\left(A+A^*-3\varphi E_n-B^{-1}AB-BA^*B^{-1}\right)\right. \\
& \left.+\left(A^*\right)^2B^{-1}+B^{-1}A^2+B^{-1}ABA^*B^{-1}-3A^*B^{-1}A\right](s). \qquad (2.16)
\end{aligned}$$

From (2.15) and (2.16) we have

$$\begin{aligned}
& -\int_x^c H(s,x)v(s)\left[C+\varphi\left(A+A^*\right)-\varphi^2B+\varphi'E_n\right](s)ds \\
= \ & -H(c,x)W(c)-\int_x^c R^{-1}(s)G_2(s,x)R^{-1}(s)ds
\end{aligned}$$

$$\begin{aligned}
&= -H(c,x)W(c) - \int_x^c R^{-1}(s)\left(Q_2^*Q_2\right)(s,x)\,R^{-1}(s)ds \\
&\quad + \int_x^c \frac{v(s)}{4}\,[h_2(x,s) \\
&\quad - H^{1/2}(x,s)\left[A^* + B^{-1}AB + 2\varphi E_n - 2\varphi B\right](s)\Big]^2 B^{-1}(s)ds \\
&\quad - \int_x^c \frac{v(s)}{4} H(s,x)\Bigg[2\varphi\left(A + A^* - B^{-1}AB - BA^*B^{-1}\right) \\
&\quad + (A^*)^2 B^{-1} + B^{-1}A^2 + B^{-1}ABA^*B^{-1} + A^*B^{-1}A\Bigg](s)ds. \qquad (2.17)
\end{aligned}$$

Set

$$\begin{aligned}
S_2(c,x) = \int_x^c &\{H(s,x)v(s)K(s) \\
&- \frac{v(s)}{4}\left[h_1(s,x)E_n + H^{1/2}(s,x)T(s)\right]^2 B^{-1}(s)\Big\}\,ds,
\end{aligned}$$

where $K(x)$ and $T(x)$ are defined by (2.5) and (2.6), respectively, then we have

$$\begin{aligned}
&\int_x^c H(s,x)v(s)K(s)ds \\
&= -H(c,x)W(c) - \int_x^c R^{-1}(s)G_2(s,x)R^{-1}(s)ds \\
&\quad + \int_x^c \frac{v(s)}{4}\left[h_1(x,s)E_n - H^{1/2}(s,x)T(s)\right]^2 B^{-1}(s)ds,
\end{aligned}$$

and so

$$S_2(c,x) = -H(c,x)W(c) - \int_x^c R^{-1}(s)G_2(s,x)R^{-1}(s)ds,$$

which yields

$$S_2(c,x) \le -H(c,x)W(c).$$

Therefore

$$\lambda_n\left[S_2(c,x)\right] \;\le\; \lambda_n\left[-H(c,x)W(c)\right] = -\lambda_1\left[H(c,x)W(c)\right]$$

$$= \quad -H(c,x)\lambda_1\left[W(c)\right]. \tag{2.18}$$

Let $x \to a^+$ in (2.18) and dividing both sides by $H(c,a)$, then we have

$$\frac{1}{H(c,a)}\lambda_n\left[S(c,a)\right] \le -\lambda_1\left[W(c)\right].$$

The proof is complete.

The following theorem is an immediate result from Lemmas 2.1 and 2.2.

Theorem 2.1. Assume that for some $c \in (a,b)$ and for some $H \in X, \varphi \in C^1[t_0,\infty)$,

$$\begin{aligned}
&\frac{1}{H(c,a)}\lambda_n\left[\int_a^c \left\{H(s,a)v(s)K(s)\right.\right.\\
&\left.\left.-\frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T\left(s\right)\right]^2 B^{-1}(s)\right\} ds\right]\\
&+\frac{1}{H(b,c)}\lambda_1\left[\int_c^b \left\{H(b,s)v(s)K(s)\right.\right.\\
&\left.\left.-\frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T\left(s\right)\right]^2 B^{-1}(s)\right\} ds\right]\\
&> 0
\end{aligned} \tag{2.19}$$

where $v(t), K(x)$ and $T(x)$ are defined as in Lemma 2.1. Then the determinant of every nontrivial prepared solution $(U(x),V(x))$ of the system (1.1) has the property that $\det U(x)$ has at least one zero in (a,b).

Proof. Suppose the contrary is true. Then without loss of generality we may assume that there is a solution $(U(x),V(x))$ of the system (1.1) such that $\det U(x) \neq 0$ for $x \in (a,b)$. From Lemmas 2.1 and 2.2 we see that both (2.4) and (2.14) hold. By adding them, we see that

$$\begin{aligned}
&\frac{1}{H(c,a)}\lambda_n\left[\int_a^c \left\{H(s,a)v(s)K(s)\right.\right.\\
&\left.\left.-\frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T\left(s\right)\right]^2 B^{-1}(s)\right\} ds\right]\\
&+\frac{1}{H(b,c)}\lambda_1\left[\int_c^b \left\{H(b,s)v(s)K(s)\right.\right.\\
&\left.\left.-\frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T\left(s\right)\right]^2 B^{-1}(s)\right\} ds\right]
\end{aligned}$$

$$\leq 0,$$

which contradicts the assumption (2.19) and completes our proof.

Theorem 2.2. If for each $x \geq x_0$, there exist $H \in X$, $\varphi \in C^1[x_0, \infty)$ and $a, b, c \in R$ such that $x \leq a < c < b$ and (2.19) holds, then the system (1.1) is oscillatory.

Proof. Pick a sequence $\{t_i\} \subset [x_0, \infty)$ such that $t_i \to \infty$ as $i \to \infty$. By our assumptions, for each $i \in N$, there exist $a_i, b_i, c_i \in R$ such that $t_i \leq a_i < c_i < b_i$ and (2.19) holds, where a, b, c are replaced by a_i, b_i, c_i, respectively. From Theorem 2.1, the determinant of every nontrivial prepared solution $(U(x), V(x))$ of the system (1.1) has the property that $\det U(x)$ has at least one zero, $x_i \in (a_i, b_i)$. Noting that $x_i > a_i \geq t_i$, $i \in N$, we see that every solution has arbitrary large zeros. Thus, every nontrivial prepared solution of the system (1.1) is oscillatory. The proof is complete.

Under a modification of the hypotheses of Theorem 2.2, we can obtain the following result.

Corollary 2.1. In Theorem 2.2, if condition (2.19) is replaced by

$$\begin{aligned}
&\frac{1}{H(c,a)} \int_a^c H(s,a)v(s)\mathrm{tr}\,(K(s))\,ds + \frac{1}{H(b,c)} \int_c^b H(b,s)v(s)\mathrm{tr}\,(K(s))\,ds \\
> \;& \frac{1}{H(c,a)} \int_a^c \frac{v(s)}{4} \mathrm{tr}\left(\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)\right) ds \\
&+ \frac{1}{H(b,c)} \int_c^b \frac{v(s)}{4} \mathrm{tr}\left(\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)\right) ds,
\end{aligned}$$

then the system (1.1) is oscillatory.

Corollary 2.2. In Theorem 2.2, if condition (2.19) is replaced by

$$\begin{aligned}
&\frac{1}{H(c,a)}\lambda_n \left[\int_a^c H(s,a)v(s)K(s)ds\right] \\
&\qquad + \frac{1}{H(b,c)}\lambda_1 \left[\int_c^b H(b,s)v(s)K(s)ds\right] \\
&> \frac{1}{H(c,a)}\lambda_1 \left[\int_a^c \frac{v(s)}{4} \left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)ds\right] \\
&+ \frac{1}{H(b,c)}\lambda_1 \left[\int_c^b \frac{v(s)}{4} \left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)ds\right], \quad (2.20)
\end{aligned}$$

then the system (1.1) is oscillatory.

Proof. Clearly, it is sufficient to prove that (2.19) holds. Note that

$$K^*(s) = K(s),$$

$$\left(\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)\right)^* = \left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)$$

and

$$\left(\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)\right)^* = \left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s).$$

Then, by Lemma 1.1, we have

$$\begin{aligned}
& \frac{1}{H(c,a)}\lambda_n\left[\int_a^c\left\{H(s,a)v(s)K(s) - \frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)\right\}ds\right] \\
& + \frac{1}{H(b,c)}\lambda_1\left[\int_c^b\left\{H(b,s)v(s)K(s) - \frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)\right\}ds\right] \\
\geq\ & \frac{1}{H(c,a)}\lambda_n\left[\int_a^c H(s,a)v(s)K(s)ds\right] \\
& + \frac{1}{H(c,a)}\lambda_n\left[-\int_a^c \frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)ds\right] \\
& + \frac{1}{H(b,c)}\lambda_1\left[\int_c^b H(b,s)v(s)K(s)ds\right] \\
& + \frac{1}{H(b,c)}\lambda_n\left[-\int_c^b \frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)ds\right] \\
=\ & \frac{1}{H(c,a)}\lambda_n\left[\int_a^c H(s,a)v(s)K(s)ds\right] \\
& - \frac{1}{H(c,a)}\lambda_1\left[\int_a^c \frac{v(s)}{4}\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s)\right]^2 B^{-1}(s)ds\right] \\
& + \frac{1}{H(b,c)}\lambda_1\left[\int_c^b H(b,s)v(s)K(s)ds\right] \\
& - \frac{1}{H(b,c)}\lambda_1\left[\int_c^b \frac{v(s)}{4}\left[h_2(b,s)E_n + H^{1/2}(b,s)T(s)\right]^2 B^{-1}(s)ds\right]
\end{aligned}$$

$> \quad 0.$

The proof is complete.

Theorem 2.3. If

$$\limsup_{x\to\infty} \lambda_n \left[\int_l^x \left\{ H(s,l)v(s)K(s) - \frac{v(s)}{4} \left[h_1(s,l)E_n - H^{1/2}(s,l)T(s) \right]^2 B^{-1}(s) \right\} ds \right] > 0, \tag{2.21}$$

$$\limsup_{x\to\infty} \lambda_1 \left[\int_l^x \left\{ H(x,s)v(s)K(s) - \frac{v(s)}{4} \left[h_2(x,s)E_n + H^{1/2}(x,s)T(s) \right]^2 B^{-1}(s) \right\} ds \right] > 0, \tag{2.22}$$

for some $H \in X$, $\varphi \in C^1[x_0,\infty)$ and for each $l \geq x_0$, where $v(t) = \exp\left(-2\mu \int_{x_0}^t \varphi(s)ds\right)$ and $K(x)$ and $T(x)$ are defined as in Lemma 2.1, then the system (1.1) is oscillatory.

Proof. For any $t \geq x_0$, let $a = t$. In (2.21) we choose $l = a$. Then there exists $c > a$ such that

$$\lambda_n \left[\int_l^x \left\{ H(s,l)v(s)K(s) - \frac{v(s)}{4} \left[h_1(s,l)E_n - H^{1/2}(s,l)T(s) \right]^2 B^{-1}(s) \right\} ds \right] > 0. \tag{2.23}$$

In (2.22) we choose $l = c$. Then there exists $b > c$ such that

$$\lambda_1 \left[\int_l^x \left\{ H(x,s)v(s)K(s) - \frac{v(s)}{4} \left[h_2(x,s)E_n + H^{1/2}(x,s)T(s) \right]^2 B^{-1}(s) \right\} ds \right] > 0. \tag{2.24}$$

Combining (2.23) and (2.24) we obtain (2.19). The conclusion thus comes from Theorem 2.2. The proof is complete.

Corollary 2.3. In Theorem 2.3, if conditions (2.21) and (2.22) are replaced by

$$\limsup_{x\to\infty} \left[\int_l^x \left\{ H(s,l)v(s)\mathrm{tr}\,(K(s)) - \frac{v(s)}{4}\mathrm{tr} \left(\left[h_1(s,a)E_n - H^{1/2}(s,a)T(s) \right]^2 B^{-1}(s) \right) \right\} ds \right] > 0,$$

$$\limsup_{x\to\infty} \left[\int_l^x \left\{ H(x,s)v(s)\mathrm{tr}\,(K(s)) - \frac{v(s)}{4}\mathrm{tr} \left(\left[h_2(x,s)E_n + H^{1/2}(x,s)T(s) \right]^2 B^{-1}(s) \right) \right\} ds \right] > 0,$$

respectively, for some $H \in X$, $\varphi \in C^1[x_0,\infty)$ and for each $l \geq x_0$, where $v(t) = \exp\left(-2\mu \int^t \varphi(s)ds\right)$ and $K(x)$ and $T(x)$ are defined as in Lemma 2.1, then the system (1.1) is oscillatory.

For the case where $H := H(t-s) \in X$, we have that $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of X containing such $H(t-s)$ is denoted by X_0. Applying Theorem 2.2 to X_0, we obtain

Theorem 2.4. If for each $t \geq t_0$, there exist $H \in X_0$, $\varphi \in [t_0\infty)$ and $a, c \in R$ such that $t \leq a < c$ and

$$\lambda_n \left[\int_a^c \left\{ H(s-a)v(s)K(s) - \frac{v(s)}{4}\left[h(s-a)E_n - H^{1/2}(s-a)T(s)\right]^2 B^{-1}(s)\right\} ds\right]$$

$$+\lambda_1 \left[\int_a^c \left(H(s-a)v(2c-s)K(2c-s)\right.\right.$$

$$\left.\left.-\frac{v(2c-s)}{4}\left[h(s-a)E_n + H^{1/2}(s-a)T(2c-s)\right]^2 B^{-1}(2c-s)\right) ds\right]$$

$$> \; 0, \tag{2.25}$$

where $v(t) = \exp\left(-2\mu \int_{x_0}^t \varphi(s)ds\right)$ and $K(x)$ and $T(x)$ are defined as in Lemma 2.1, then the system (1.1) is oscillatory.

Proof. Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $w \in L[a, b]$, we have

$$\int_c^b w(s)ds = \int_a^c w(2c-s)ds.$$

Hence

$$\int_c^b H(b-s)v(s)K(s)ds = \int_a^c H(s-a)v(2c-s)K(2c-s)ds$$

and

$$\int_c^b \frac{v(s)}{4}\left[h(b-s)E_n + H^{1/2}(b-s)T(s)\right]^2 B^{-1}(s)ds$$

$$= \int_c^a \frac{v(2c-s)}{4}\left[h(b-s)E_n + H^{1/2}(b-s)T(2c-s)\right]^2 B^{-1}(2c-s)ds.$$

Thus (2.25) implies that (2.19) holds for $H \in X_0$, and $\varphi \in C^1[x_0, \infty)$ and therefore the system (1.1) is oscillatory by Theorem 2.2. The proof is complete.

By Theorem 2.4 and Lemma 1.1, we have the following result.

Corollary 2.4. In Theorem 2.4, if condition (2.25) is replaced by

$$\lambda_n \left[\int_a^c H(s-a)v(s)K(s)ds\right] + \lambda_1 \left[\int_c^b H(s-a)v(2c-s)K(2c-s)ds\right]$$

$$
\begin{aligned}
> & \frac{1}{4}\lambda_1 \left[\int_a^c v(s) \left[h(s-a)E_n - H^{1/2}(s-a)T(s) \right]^2 B^{-1}(s)ds \right] \\
& \frac{1}{4}\lambda_1 \left[\int_c^b v(2c-s) \left[h(s-a)E_n + H^{1/2}(s-a)T(2c-s) \right]^2 \right. \\
& \left. \times B^{-1}(2c-s)ds \right], \qquad (2.26)
\end{aligned}
$$

then the system (1.1) is oscillatory.

From the above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $H(t,s)$ and φ. For instance, let us consider the function H defined by

$$H(x,s) = (x-s)^{\lambda}, \quad x \geq s \geq x_0,$$

where $\lambda > 1$ is a constant. By Theorem 2.4, we obtain

Corollary 2.5. The system (1.1) is oscillatory provided that for each $l \geq x_0$ and for some $\lambda > 1$, the following two inequalities hold:

$$
\begin{aligned}
\limsup_{t\to\infty} \lambda_n & \left[\frac{1}{t^{\lambda-1}} \int_l^t (s-l)^{\lambda} v(s) \left(K(s) - \frac{1}{4} \left(\frac{\lambda}{(s-l)} E_n - T(s) \right)^2 \right) \right. \\
& \left. \times B^{-1}(s)ds \right] > 0, \qquad (2.27)
\end{aligned}
$$

and

$$
\begin{aligned}
\limsup_{t\to\infty} \lambda_1 & \left[\frac{1}{t^{\lambda-1}} \int_l^t (t-s)^{\lambda} v(s) \left(K(s) - \frac{1}{4} \left(\frac{\lambda}{(t-s)} E_n + T(s) \right)^2 \right) \right. \\
& \left. \times B^{-1}(s)ds \right] > 0, \qquad (2.28)
\end{aligned}
$$

where $v(t) = \exp\left(-2\mu \int_{x_0}^t \varphi(s)ds\right)$ and $K(x)$ and $T(x)$ are defined as in Lemma 2.1.

If we take $\varphi \equiv 0, B(x) = E_n$ and $A(x) \equiv 0$, then $T(x) = 0$ and $K(x) = -C(x)$. By Corollary 2.5, we have

Theorem 2.5. Assume that $B(x) = E_n$ and $A(x) \equiv 0$. Then the system (1.1) is oscillatory provided that for each $l \geq x_0$ and for some $\lambda > 1$, the following two inequalities hold:

$$\limsup_{x\to\infty} \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^{\lambda} \left(-C(s)\right) ds \right] > \frac{\lambda^2}{4(\lambda-1)} \qquad (2.29)$$

and

$$\limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] > \frac{\lambda^2}{4(\lambda-1)}. \tag{2.30}$$

Proof. Let

$$H(x-s) = (x-s)^\lambda, x \geq s \geq x_0, \lambda > 1.$$

Then

$$h(x-s) = \lambda (x-s)^{(\lambda-2)/2}.$$

Noting that

$$\int_l^x h^2(s-l) ds = \int_l^x \lambda^2 (s-l)^{\lambda-2} ds = \frac{\lambda^2}{\lambda-1} (x-l)^{\lambda-1},$$

and

$$\int_l^x h^2(x-s) ds = \int_l^x \lambda^2 (x-s)^{\lambda-2} ds = \frac{\lambda^2}{\lambda-1} (x-l)^{\lambda-1}.$$

Hence,

$$\lim_{t\to\infty} \frac{1}{4x^{\lambda-1}} \int_l^x h^2(s-l) ds = \frac{\lambda^2}{4(\lambda-1)}, \tag{2.31}$$

and

$$\lim_{t\to\infty} \frac{1}{4x^{\lambda-1}} \int_l^x h^2(x-s) ds = \frac{\lambda^2}{4(\lambda-1)}. \tag{2.32}$$

From (2.29) and (2.31) and (2.30) and (2.32), we have by Lemma 1.1

$$\begin{aligned} & \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x \left\{ (s-l)^\lambda \left(-C(s)\right) - \frac{1}{4} h^2(s-l) E_n \right\} ds \right] \\ \geq \; & \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^\lambda \left(-C(s)\right) ds \right] \\ & + \lambda_n \left[-\frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(s-l) E_n ds \right] \\ = \; & \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^\lambda \left(-C(s)\right) ds \right] \\ & - \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(s-l) E_n ds \right], \end{aligned}$$

and

$$\begin{aligned}
& \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \left\{ (x-s)^\lambda \left(-C(s)\right) - \frac{1}{4} h^2(x-s) E_n \right\} ds \right] \\
\geq\ & \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] \\
& + \lambda_n \left[- \frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(x-s) E_n ds \right] \\
=\ & \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] \\
& - \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(x-s) E_n ds \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
& \limsup_{x\to\infty} \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x \left\{ (s-l)^\lambda \left(-C(s)\right) - \frac{1}{4} h^2(s-l) E_n \right\} ds \right] \\
\geq\ & \limsup_{x\to\infty} \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^\lambda \left(-C(s)\right) ds \right] \\
& - \limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(s-l) E_n ds \right] \\
=\ & \limsup_{x\to\infty} \lambda_n \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^\lambda \left(-C(s)\right) ds \right] - \frac{\lambda^2}{4(\lambda-1)} > 0,
\end{aligned}$$

and

$$\begin{aligned}
& \limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \left\{ (x-s)^\lambda \left(-C(s)\right) - \frac{1}{4} h^2(x-s) E_n \right\} ds \right] \\
\geq\ & \limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] \\
& - \limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x \frac{1}{4} h^2(x-s) E_n ds \right] \\
=\ & \limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] - \frac{\lambda^2}{4(\lambda-1)} > 0.
\end{aligned}$$

By Corollary 2.5, the conclusion is true. The proof is complete.

Remark 2.1. The results in this paper are presented in a form with high degree of generality, and thus they give many possibilities for oscillation criteria with an appropriate choice of the functions H and φ. For example, there are interesting consequences if we apply Theorems 2.1-2.5 with

$$H(t,s)=\left[\int_s^t \frac{dx}{\Phi(x)}\right]^n, \quad t\geq s,$$

where $n>1$ is a constant and Φ is a positive continuous function on $[x_0,\infty)$ such that

$$\int_{x_0}^{\infty}\frac{dx}{\Phi(x)}=\infty.$$

An important case to be considered is $\Phi(x)=x^{\beta}$ with β real number.

3. Examples

The following example illustrate Theorem 2.4. It is clear that none of the results in [1, 2-41] are applicable since

$$\int^{\infty} c_i(x)dx=-\infty,\ i=1,2,$$

where $C(x)=\operatorname{diag}(-c_1(x),-c_2(x))$.

Example 3.1. Consider the four-dimensional system (1.1) where

$$A(x)=\frac{1}{2}E_2, B(x)=E_2 \text{ and } C(x)=\operatorname{diag}(-c_1(x),-c_2(x)),$$

where

$$c_1(x)=\begin{cases} 5(s-3n), & 3n\leq x\leq 3n+1,\\ 5(-x+3n+2), & 3n+1<x\leq 3n+2,\\ -n & 3n+2<x\leq 3n+3,\end{cases}$$

and

$$c_2(x)=\begin{cases} 5(s-3n), & 3n\leq x\leq 3n+1,\\ 5(-x+3n+2), & 3n+1<x\leq 3n+2,\\ -n^2 & 3n+2<x\leq 3n+3,\end{cases}$$

$n\in\{0,1,2,...\}=N_0$. Then

$$T(x)=E_2 \text{ and } K(x)=-C(x).$$

For any $t > 0$ there exists $n \in N_0$ such that $3n > t$. Let $a = 3n, c = 3n + 1$ and $H(x - s) = (x - s)^2$. Choose $\varphi(x) = 0$, then $v(x) = 1$. Therefore,

$$\lambda_2 \left[\int_a^c \left\{ (s-a)^\lambda K(s) - \frac{1}{4}\left[h(s-a)E_2 - H^{1/2}(s-a)E_2 \right]^2 \right\} ds \right]$$
$$+ \lambda_1 \left[\int_a^c \left\{ H(s-a)K(2c-s) - \frac{1}{4}\left[h(s-a)E_2 + H^{1/2}(s-a)E_2 \right]^2 \right\} ds \right]$$
$$= \frac{5}{4} + \frac{1}{12} - \frac{8}{12} - \frac{5}{4} + \frac{10}{3} - \frac{27}{12} + \frac{8}{12} = \frac{7}{6} > 0.$$

It is easy to see that (2.25) holds and hence the system (1.1) is oscillatory by Theorem 2.4. Note that in this equation we have $\int_0^\infty (-c_i(s))\,ds = \infty$ for $i = 1, 2$. However, the oscillation cannot be demonstrated either by known criteria in [1, 2-41] or other criteria in the literature.

Example 3.2. Consider the Euler differential system

$$U''(x) + \operatorname{diag}\left(\frac{\alpha}{t^2}, \frac{\beta}{t^2} \right) U(x) = 0, \quad x \geq 1, \tag{3.1}$$

where $\alpha \geq \beta > 0$. In the corresponding Hamiltonian system to (3.1), $A(x) \equiv 0, B(x) = E_2$ and

$$C(x) = -\operatorname{diag}\left(\frac{\alpha}{t^2}, \frac{\beta}{t^2} \right), x \geq 1.$$

Choose $\varphi(x) \equiv 0$, then $v(x) = 1, T(x) \equiv 0$ and $K(x) = -C(x)$. Therefore, for $\lambda > 1$,

$$\limsup_{x\to\infty} \lambda_1 \left[\frac{1}{x^{\lambda-1}} \int_l^x (x-s)^\lambda \left(-C(s)\right) ds \right] = \frac{\alpha}{\lambda - 1},$$

and

$$\limsup_{x\to\infty} \lambda_2 \left[\frac{1}{x^{\lambda-1}} \int_l^x (s-l)^\lambda \left(-C(s)\right) ds \right] = \frac{\beta}{\lambda - 1}.$$

Hence, for any $\beta > 1/4$ there exists $\lambda > 1$ such that $\beta/(\lambda-1) > \lambda^2/4(\lambda-1)$. This means that (2.29) and (2.30) hold for the same λ. Applying Theorem 2.5, we find that (3.1) is oscillatory for $\beta > 1/4$.

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