International Journal of Differential Equations and Applications
Volume 10 No. 3 2005, 283-293

APPLICATION OF ADOMIAN METHOD TO NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

M. Saleh[1], M. EL-Saeidy[2], I. EL-Kalla[3 §], D. Hammad[4]
[1,2,3,4]Department of Engineering and Mathematics
Faculty of Engineering
Mansura University
Mansura, EGYPT
[1]e-mail: mostafasaleh@mans.edu.eg
[3]e-mail: al_kalla@mans.edu.eg

Abstract: Adomian Decomposition Method (ADM) is of a great important to mathematicians nowadays. In this paper, ADM and Kutta-4-Method were applied to nonlinear systems of ordinary differential equations. From a comparison study, it was found that ADM offers a simple and more accurate approximate solution. Also, a higher-order nonlinear ordinary differential equation was transformed to the corresponding system of equations. Both equation and corresponding system were solved. A comparison study using ADM shows that, this transformation simplifies calculations.

AMS Subject Classification: —???—
Key Words: ordinary differential equations, nonlinear systems, higher-order nonlinear ordinary differential equation, Kutta-4-Method, Adomian decomposition method,

1. Introduction

The decomposition method was introduced by Adomian, since the beginning of the 1980's [4, 2]. ADM solves successfully different types of equations (algebraic [3, 1], differential [4, 2], integral and integro-differential equations

Received: April 14, 2006

[§]Correspondence author

[10] in both linear and non-linear cases. ADM provides immediate and visible symbolic terms of analytic solution with high accuracy and provides the solution in rapid convergent series with elegantly compatible terms. ADM is implemented to drive analytic approximate solution in terms of the decomposition series to general systems of ordinary differential equation [11, 9, 6]. Bellomo and Surafy an [5, 6], Bellomo and Monaco [12, 6] and Wazwaz [8, 6] implemented the method is a comparative way. Recently, ADM transform the systems of ordinary differential equation in to a set of recursive relations the resulting system of recursive relations can be easily handled. It is also quite straight forward to write computer codes in any symbolic languages [10]. This paper investigates the performance of ADM applied to nonlinear system of ordinary differential equation with constant or variable coefficients. A comparison study with Kutta-4-Method show that ADM offer a simple and more accurately technique. A higher-order nonlinear ordinary differential equation was transformed to the corresponding system of equations. Both equation and corresponding system were solved. A comparison study using ADM show that this transformation simplify calculations.

2. The General Algorithm

In this section we are going to apply the ADM on system of nonlinear ordinary differential equations in which the i-th equation takes the form

$$\pounds u_i\left(t\right)=\alpha_i\left(t\right)f_i\left(u_1,u_2,...,u_n\right)+g_i\left(t\right),\quad i=1,2,...,m\,, \tag{1}$$

where f_i are nonlinear terms which are assumed to be analytic functions, g_i and α_i are known functions, and we are seeking the solution u_i satisfying (1). We assume that for every g_i equation (1) has one and only one solution.

Applying ADM as in [4], [2], [3], [1], equation(1) subjected to suitable initial conditions where $\pounds$ is the n-th order linear ordinary differential operator $\pounds = L + R$, $L = \frac{d^n}{dt^n}$, and $R = \sum\limits_{k=0}^{n-1} a_k\left(t\right)\frac{d^k}{dt^k}$ substitute in (1) we get,

$$\left(L+R\right)u_i=\alpha_i\left(t\right)f_i\left(u_1,u_2,...,u_n\right)+g_i\left(t\right)\,. \tag{2}$$

Operating on both sides of equation (2) with the inverse operator of L namely

L^{-1}, where $L^{-1} = \int_0^t ...k\text{-times}... \int_0^t dt...dt$ yields,

$$u_i(t) = \theta_i + L^{-1}\alpha_i(t) f_i(u_1, u_2, \ldots, u_n) - L^{-1}Ru_i + L^{-1}g_i(t), \qquad i = 1, 2, \ldots, m, \tag{3}$$

where, $\theta_i = \theta_i(t)$ are the solutions of $L\theta_i = 0$ [9], satisfies the given initial conditions.

Adomian solution of (3) is an infinite series,

$$u_i = \sum_{n=0}^{\infty} u_{i,n}, \quad i = 1, 2, ..., m, \tag{4}$$

and the nonlinear term f_i is

$$f_i(u_1, u_2, ..., u_n) = \sum_{n=0}^{\infty} A_{i,n}, \quad i = 1, 2, ..., m, \tag{5}$$

where $A_{i,n}$ are polynomials (called Adomian polynomials) of $u_0, ..., u_n$ (see [4], [2], [3], [1], [10]) given by

$$A_{i,n} = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)\left[f_i\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{6}$$

Substituting (4) and (5) into (1) yields

$$\sum_{n=0}^{\infty} u_{i,n} = \theta_i - L^{-1}Ru_i + L^{-1}\sum_{n=0}^{\infty} A_{i,n} + L^{-1}g_i.$$

Setting

$$u_{i,0} = \theta_i + L^{-1}g_i,$$

then

$$u_{i,n+1} = -L^{-1}Ru_i + L^{-1}A_{i,n}(u_0, u_1, ...u_n), \qquad i = 1, 2, ..., \quad n = 0, 1, 2, ... \tag{7}$$

Thus all components of u_i can be calculated once the $A_{i,n}$ are given, finally the solution of the system is $u_i = \sum_{n=0}^{\infty} u_{i,n}$

3. Numerical Examples

To give a clear overview of the ADM as a numerical tool, we apply the Kutta-4-Method [8] and Adomian decomposition methods on many examples, so that the comparison is made numerically.

Example 1. Consider nonlinear system with constant coefficient

$$\begin{aligned} \dot{u}_1(t) &= u_1(t) + u_2^2(t) , \\ \dot{u}_2(t) &= 2u_2(t) - e^t , \end{aligned} \tag{8}$$

and initial conditions $u_1(0) = 1$ and $u_2(0) = 1$ whose exact solution is $u_{E1}(t) = e^{2t}$ and $u_{E2}(t) = e^t$.

Operating with L^{-1}on (8) we obtain

$$\begin{aligned} u_1 &= u_1(0) - L^{-1}u_1 + L^{-1}N_1 , \\ u_2 &= u_2(0) + 2L^{-1}u_1 - L^{-1}\left(e^t\right) . \end{aligned}$$

For the nonlinear term

$$N_1 = u_2^2 = \sum_{n=0}^{\infty} A_{1,n} .$$

To derive the solution by using decomposition method, follows the recursive relationship (7)

$$\begin{aligned} u_{1,0} &= 1 , \\ u_{1,n+1} &= L^{-1}u_{1,n} + L^{-1}A_{1,n} , \\ u_{2,0} &= 1 + L^{-1}\left(e^t\right) , \\ u_{2,n+1} &= 2L^{-1}u_{2,n} , \quad n = 0, 1, 2, \ldots . \end{aligned}$$

To accelerate the convergence of the solution we use the expansion of $\exp(t) = \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots\right)$ to minimize the size of calculations using *Matlab 7*, ADM and Kutta-4-Method algorithms were executed and the following results of absolute errors were obtained in Table 1(a) and Table 1(b), where decomposition series solution $u_i(t)$, Kutta-4-Method solution $u_{Ri}(t)$ and exact solution $u_{Ei}(t)$ in the same interval $0 \leq t \leq 1$ and number of terms $N = 10$

It is clear from the comparison that the decomposition solution is much closer to the exact solution and also is easier to use than Kutta-4-Method solution.

t	$\|u_{E1}(t) - u_1(t)\|$	$\|u_{E1}(t) - u_{R1}(t)\|$
0.0	0.00000E+00	0.00000E+00
0.2	1.92450E-09	3.09500E-06
0.4	7.50073E-09	9.65632E-06
0.6	1.38543E-09	2.28541E-05
0.8	2.02108E-08	4.85931E-05
1.0	1.98410E-08	9.78168E-05

Table 1(a):

t	$\|u_{E2}(t) - u_2(t)\|$	$\|u_{E2}(t) - u_{R2}(t)\|$
0.0	0.00000E+00	0.00000E+00
0.2	1.24544E-10	1.71579E-06
0.4	2.40055E-10	4.65532E-06
0.6	1.29625E-10	9.50455E-06
0.8	8.64071E-10	1.73054E-05
1.0	4.20060E-11	2.96351E-05

Table 1(b):

Example 2. Consider nonlinear system with variable coefficient

$$\begin{aligned} \dot{u}_1(t) &= 2u_2^2(t) + e^{-t}u_1(t) - u_2(t)\ , \\ \dot{u}_2(t) &= e^{-t}u_1(t) - u_3(t) + tu_2(t)\ , \\ \dot{u}_3(t) &= u_2(t) + u_3(t)\ , \end{aligned} \tag{9}$$

and initial conditions $.u_1(0) = 1$, $u_2(0) = 1$ and $u_2(0) = 0$ whose exact solution is $u_{E1}(t) = e^{2t}$, $u_{E2}(t) = e^t$ and $u_{E3}(t) = te^t$.

Operating with L^{-1} on (9) we obtain

$$\begin{aligned} u_1 &= u_1(0) - L^{-1}(e^{-t}u_1(t) - u_2(t)) + 2L^{-1}N_1\,, \\ u_2 &= u_2(0) + L^{-1}(e^{-t}u_1(t) - u_3(t) + tu_2(t))\,, \\ u_3 &= L^{-1}(u_2(t) + u_3(t))\,. \end{aligned}$$

For the nonlinear term

$$N_1 = u_2^2 = \sum_{n=0}^{\infty} A_{1,n}\,.$$

t	$\|u_{E1}(t) - u_1(t)\|$	$\|u_{E1}(t) - u_{R1}(t)\|$
0.0	0.00000E+00	0.00000E+00
0.2	1.09596E-07	4.00999E-06
0.4	2.32946E-06	3.49256E-05
0.6	7.69828E-05	1.91017E-04
0.8	1.23881E-03	6.51507E-03
1.0	9.92281E-03	1.76464E-03

Table 2(a):

t	$\|u_{E2}(t) - u_2(t)\|$	$\|u_{E2}(t) - u_{R2}(t)\|$
0.0	1.11759E-08	0.00000E+00
0.2	2.43491E-09	1.33989E-06
0.4	1.78752E-07	7.53888E-05
0.6	8.89998E-06	2.12243E-05
0.8	1.33071E-04	4.70919E-04
1.0	1.00673E-03	9.28110E-03

Table 2(b):

To derive the solution by using decomposition method, following the recursive relationship (7)

$$
\begin{aligned}
u_{1,0} &= 1\,, \\
u_{1,n+1} &= 2L^{-1}\left(A_{1,n}\right) + L^{-1}\left(e^{-t}u_{1,n} - u_{2,n}\right)\,, \\
u_{2,0} &= 1\,, \\
u_{2,n+1} &= L^{-1}\left(2e^{-t}u_{1,n} - 3u_{3,n} + tu_{2,n}\right)\,, \\
u_{3,0} &= 0\,, \\
u_{3,n+1} &= L^{-1}\left(u_{2,n} + u_{3,n}\right)\,.
\end{aligned}
$$

Like in Example 1, by expanding $\exp(-t) = \left(1 - \frac{t}{1!} + \frac{t^2}{2!} - ...\right)$, using *Matlab* 7, ADM and Kutta-4-Method algorithms were executed and the following results of absolute errors were obtained in Table 2(a), Table 2(b) and Table 2(c), where decomposition series solution $u_i(t)$, Kutta-4-Method solution $u_{Ri}(t)$ and exact solution $u_{Ei}(t)$ in the same interval $0 \leq t \leq 1$ and number of terms $N = 10$.

It is clear from the comparison that the decomposition solution much closer to the exact solution and also easier to use than Kutta-4-Method solution

t	$\lVert u_{E3}(t) - u_3(t) \rVert$	$\lVert u_{E3}(t) - u_{R3}(t) \rVert$
0.0	2.86003E-09	0.00000E+00
0.2	1.17183E-08	2.76634E-06
0.4	2.22812E-08	6.09146E-05
0.6	2.33364E-06	1.02478E-04
0.8	3.86635E-05	2.55697E-04
1.0	3.22911E-04	4.24608E-04

Table 2(c):

3.1. Transforming a Higher Order ODE to the Corresponding First Order System

We use ADM to solve a higher order ODE subjected to initial conditions and converting it to corresponding system of first order equations

To convert a general m-th order ordinary differential equation of the form

$$u^{(m)}(t) = f\left(t, u, \dot{u}, \ddot{u}, ..., u^{(m-1)}\right), \quad a \le t \le b,$$

and initial condition $u(a) = \alpha_1$, $\dot{u}(a) = \alpha_2$,..., $u^{(m-1)}(a) = \alpha_m$, into a system of first order equations [7] in the form

$$\dot{u}_i(t) = f_i(t, u_1, u_2, ..., u_m), \qquad i = 0, 1, 2, .., m. \tag{10}$$

Let $u_1(t) = u(t)$, $u_2(t) = \dot{u}(t)$, $u_3(t) = \ddot{u}(t)$, ..., $u_m(t) = u^{(m-1)}(t)$, using this notation, we obtain first order system

$$\begin{aligned}
\dot{u}_1(t) &= u_2, \\
\dot{u}_2(t) &= u_3, \\
&\vdots \\
\dot{u}_{m-1}(t) &= u_m, \\
\dot{u}_m(t) &= u^{(m)} = f\left(t, u, \dot{u}, \ddot{u}, ..., u^{(m-1)}\right) = f(t, u_1, u_2, ..., u_m),
\end{aligned}$$

with initial condition $u_1(a) = u(a) = \alpha_1$, $u_2(a) = \dot{u}(a) = \alpha_2$,..., $u_m(a) = u^{(m-1)}(a) = \alpha_m$

Example 3. Consider the following nonlinear 4-th order ordinary differential equation

$$u^{(iv)} = 2\frac{e^{-t}}{t^2}(\ddot{u})^2 + \dddot{u} - e^t, \tag{11}$$

and initial condition $u_1(0) = 1$, $\dot{u}(0) = 1$, $\ddot{u}(0) = 2$, and $\dddot{u}(0) = 3$ whose exact solution is $u_E(t) = te^t$

Operating with L^{-1}on (11) we obtain

$$u = u(0) + L^{-1}\left(2\frac{e^{-t}}{t^2}N_1\right) + L^{-1}\left(\dddot{u}\right) - L^{-1}e^t\,,$$

where $L^{-1} = \int_0^t\int_0^t\int_0^t\int_0^t ()\,dt...dt$

For the nonlinear term

$$N_1 = (\ddot{u})^2 = \sum_{n=0}^{\infty} A_n\,.$$

To derive the solution by using decomposition method, follows the recursive relationship (7)

$$\begin{aligned}\theta &= \frac{t^3}{2} + t^2 + t\,,\\ u_0 &= -L^{-1}\left(e^t\right) + \frac{t^3}{2} + t^2 + t\,,\\ u_{n+1} &= L^{-1}(\dddot{u}_n(t)) + L^{-1}\left(2\frac{e^{-t}}{t^2}A_n\right)\,.\end{aligned}$$

Let $u_1(t) = u(t)$, $u_2(t) = \dot{u}(t)$, $u_3(t) = \ddot{u}(t)$ and $u_4(t) = \dddot{u}(t)$, with initial conditions $u_1(0) = 0$, $u_2(0) = 1$, $u_3(0) = 2$, and $u_4(0) = 3$ whose exact solution is $u_{E1}(t) = te^t$, $u_{E2}(t) = e^t(1+t)$, $u_{E3}(t) = e^t(2+t)$, and $u_{E4}(t) = e^t(3+t)$.

We can convert (11) into the corresponding nonlinear system of differential equation (12)

$$\begin{aligned}\dot{u}_1(t) &= u_2(t)\,,\\ \dot{u}_2(t) &= u_3(t)\,,\\ \dot{u}_3(t) &= u_4(t)\,,\\ \dot{u}_4(t) &= \frac{2e^{-t}}{t^2}u_1^2(t) + u_4(t) - e^t\,.\end{aligned} \tag{12}$$

t	$\|u_E(t) - u_S(t)\|$	$\|u_E(t) - u(t)\|$
0.0	0.00000E+00	0.00000E+00
0.2	4.81754E-05	3.10137E-09
0.4	8.36278E-04	4.85166E-07
0.6	4.59470E-03	1.01534E-05
0.8	1.57512E-02	9.34586E-05
1.0	4.16363E-02	5.49591E-04

Table 3:

Operating with L^{-1}on system (12) we obtain

$$\begin{aligned}
u_1(t) &= u_1(0) + L^{-1}u_2(t) , \\
u_2(t) &= u_2(0) + L^{-1}u_3(t) , \\
u_3(t) &= u_3(0) + L^{-1}u_4(t) , \\
u_4(t) &= u_4(0) + L^{-1}\frac{2e^{-t}}{t^2}N_1 + L^{-1}u_4(t) - L^{-1}e^t .
\end{aligned}$$

To derive the solution by using decomposition method, follows the recursive relationship (7)

$$\begin{aligned}
u_{1,0} &= 0 , \\
u_{1,n+1} &= L^{-1}u_{2,n}(t) , \\
u_{2,0} &= 1 , \\
u_{2,n+1} &= \quad L^{-1}u_{3,n}(t) , \\
.u_{3,0} &= 2 , \\
u_{3,n+1} &= L^{-1}u_{4,n}(t) , \\
u_{4,0} &= 3 - L^{-1}e^t , \\
u_{4,n+1} &= L^{-1}\frac{2e^{-t}}{t^2}A_n + L^{-1}u_{4,n}(t) .
\end{aligned}$$

Using *Matlab* 7 and ADM algorithm were executed and the following results of absolute errors were obtained in Table 3, where decomposition series solution of one equation $u(t)$, decomposition series solution of corresponding system of differential equations $u_s(t)$ and exact solution $u_E(t)$ in the same interval $0 \leq t \leq 1$ and number of terms $N = 10$.

4. Conclusion

ADM is a powerful device for solving system of ordinary differential equation. From a comparison study between ADM and Kutta-4-Method, it was found that ADM offers a simple and more accurate approximate solution. A higher-order nonlinear ordinary differential equation was transformed to the corresponding system of equations. Both equation and corresponding system were solved. A comparison study using ADM shows that this transformation simplifies calculations.

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