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UNIQUENESS OF POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN TIME SCALES

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Abstract: Sufficient conditions for the uniqueness of positive solutions for nonlinear boundary value problems in time scales

$$\text{(BVP)} \qquad \begin{cases} (E) \quad (r(t)u^{\Delta})^{\Delta} + f(t, u^{\sigma}(t)) = 0, \quad \text{in} \quad (\theta_1, \theta_2), \\ (BC) \begin{cases} \alpha_1 u(\theta_1) - \beta_1 u^{\Delta}(\theta_1) = 0, \\ \alpha_2 u(\sigma^2(\theta_2)) + \beta_2 u^{\Delta}(\sigma(\theta_2)) = 0 \end{cases} \end{cases}$$

are established,where $\alpha_i, \beta_i \geq 0$ and ${\alpha_i}^2 + {\beta_i}^2 \neq 0 \quad (i = 1, 2)$.

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1. Introduction

In this paper, we are concerned with the uniqueness of positive solutions for nonlinear boundary value problems in time scales

$$(E) \qquad (r(t)u^{\Delta})^{\Delta} + f(t, u^{\sigma}(t)) = 0, \quad \theta_1 < t < \theta_2,$$

subject to one of the following sets of boundary conditions:

$$(BC.1) \qquad u(\theta_1) = \xi_1 \geq 0, \quad u(\sigma^2(\theta_2)) = \xi_2 \geq 0,$$

$$(BC.2) \qquad u(\theta_1) = \xi_1 \geq 0, \quad u^{\Delta}(\sigma(\theta_2)) = \xi_2 \geq 0,$$

$$(BC.3) \qquad u^{\Delta}(\theta_1) = \xi_1 \leq 0, \quad u(\sigma^2(\theta_2)) = \xi_2 \geq 0,$$

where $(\theta_1, \theta_2) \subseteq (-\infty, \infty)$, $r(t) \geq 0$ on $[\theta_1, \sigma(\theta_2)]$ and $f : (\theta_1, \sigma(\theta_2)) \times$

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$(0, \infty) \to (0, \infty)$ satisfy the following condition:
(H) $f(t, x)$ is locally Lipschitz continuous for x in $(0, \infty)$, $f(t, x)/x$ is strictly decreasing w.r.t. $x \in (0, \infty)$ for each fixed $t \in (\theta_1, \sigma(\theta_2))$.

Furthermore, we use the uniqueness theorems of (E) with respect to the boundary conditions (BC.i)(i=1,2,3) to show that

$$\text{(BVP)} \qquad \begin{cases} (E) \ \ (r(t)u^{\Delta})^{\Delta} + f(t, u^{\sigma}(t)) = 0, \qquad \text{in} \qquad (\theta_1, \theta_2), \\ (BC) \ \begin{cases} \alpha_1 u(\theta_1) - \beta_1 u^{\Delta}(\theta_1) = 0, \\ \alpha_2 u(\sigma^2(\theta_2)) + \beta_2 u^{\Delta}(\sigma(\theta_2)) = 0 \end{cases} \end{cases}$$

has at most one positive solution in $C^1([\theta_1, \theta_2])$, where $\alpha_i, \beta_i \geq 0$ and ${\alpha_i}^2 + {\beta_i}^2 \neq 0 \qquad (i = 1, 2)$.

Now, we briefly introduce the time scales theory and refer to Bohner and Peterson [3] for further details.

Definition 1.A. Let $\mathbf{T}$ be a closed subset of the real numbers $\mathbf{R}$ with the property that

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbf{T}\} \in \mathbf{T},$$

and

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbf{T}\} \in \mathbf{T},$$

for all $t \in \mathbf{T}$. We assume thoughout that $\mathbf{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbf{R}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$ we say t is right dense, while if $\rho(t) = t$ we say t is left dense.
Thoughout this paper we make the blanket assumption that $a \leq b$ are points in $\mathbf{T}$.

Definition 1.B. Define the interval in $\mathbf{T}$

$$[a, b] := \{t \in \mathbf{T} \quad \text{such that} \quad \mathbf{a} \leq \mathbf{t} \leq \mathbf{b}\}.$$

Other types of intervals are defined similarly.

Definition 1.C. Assume $x : \mathbf{T} \to \mathbf{R}$ and fix $t \in \mathbf{T}$, then we define $x^{\Delta}(\mathrm{t})$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the **delta derivative** of $x(t)$.
It can be shown that if $x : \mathbf{T} \to \mathbf{R}$ is continuous at $t \in \mathbf{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition 1.D. If $F^{\Delta}(t) = f(t)$, then we define an integral by

$$\int_a^t f(\tau) \ \ \Delta\tau = F(t) - F(a).$$

The uniqueness problem concerning (E) has been studied by many authors in continuous case., i.e: $\mathbf{T} = \mathbf{R}$.For example, Gatica, Oliker and Waltman [8], Kwong [16], Dalmasso [5,6],Brezis and Oswald [4], Krasnoselskii [15],Wong [24] and the excellent book by Agarwal and Lakshmikantham [1].

Remark. The hypothesis "$f(t, x)$ is **locally Lipschitz continuous** for x in $(0, \infty)$",which guarantee the uniqueness of positive solution of (E) with respect to **non-zero initial condition** :$u(\eta) \neq 0$ and $u^{\Delta}(\eta) \in \mathbf{R}$ for each given $\eta \in [\theta_1, \sigma(\theta_2)]$. Here, we shall point out that this condition can be replace by another condition, if it can guarantee the uniqueness of positive solution of (E) with respect to **non-zero initial condition** :$u(\eta) \neq 0$ and $u^{\Delta}(\eta) \in \mathbf{R}$ for each given $\eta \in [\theta_1, \sigma(\theta_2)]$.

2. Main Results

Let u and v be two distinct positive solutions of (E)on $[\theta_1, \sigma^2(\theta_2)]$. Throughout this paper, we define

$$\begin{aligned} w(t) \ &= u(t)(r(t)v^{\Delta}(t)) - v(t)(r(t)u^{\Delta}(t)) \\ &= u^{\sigma}(t)(r(t)v^{\Delta}(t)) - v^{\sigma}(t)(r(t)u^{\Delta}(t)) \qquad (1) \end{aligned}$$

for $t \in [\theta_1, \sigma(\theta_2)]$ (cf :Lemma 4.28 of Bohner and Peterson [3]).

It is clear that $w(t)$ satisfies

$$\begin{aligned} w^{\Delta}(t) \ &= \{u^{\sigma}(rv^{\Delta})^{\Delta} - v^{\sigma}(ru^{\Delta})^{\Delta}\}(t) \\ &= u^{\sigma}(t)v^{\sigma}(t)\{\tfrac{f(t,u^{\sigma}(t))}{u^{\sigma}(t)} - \tfrac{f(t,v^{\sigma}(t))}{v^{\sigma}(t)}\} \qquad (2) \end{aligned}$$

for $t \in (\theta_1, \sigma(\theta_2))$(cf :Theorem 1.20 of Bohner and Peterson [3]). Follows from the hypothesis (H) and the above-mentioned results, we have the following:

Lemma 2.1. Let u and v be distinct positive solutions of (E) on $[\theta_1, \sigma^2(\theta_2)]$ and $u > v$ in $(\theta_1, \sigma^2(\theta_2))$. Then $w^{\Delta}(t) < 0$ in $(\theta_1, \sigma^2(\theta_2))$, that is, $w(t)$ is strictly decreasing on $[\theta_1, \sigma^2(\theta_2)]$.

Theorem 2.2. Suppose that there is no $t \in (\theta_1, \sigma^2(\theta_2))$ such that t is left-dense and right-scattered, simultaneously. Then, the boundary value problem (E)-(BC.1) has at most one positive solution on $[\theta_1, \sigma^2(\theta_2)]$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)-(BC.1)on $[\theta_1, \sigma^2(\theta_2)]$. We claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. If $v^\Delta(\theta_1) > u^\Delta(\theta_1)$ or $v^\Delta(\sigma(\theta_2)) < u^\Delta(\sigma(\theta_2))$, then

$$\begin{aligned} v(\sigma(\theta_1)) &= v(\theta_1) + (\sigma(\theta_1) - \theta_1)v^\Delta(\theta_1) \\ &= u(\theta_1) + (\sigma(\theta_1) - \theta)v^\Delta(\theta_1) \\ &> u(\theta_1) + (\sigma(\theta_1) - \theta)u^\Delta(\theta_1) = u(\sigma(\theta_1)) \end{aligned} \tag{3}$$

or

$$\begin{aligned} v(\sigma^2(\theta_2)) &= v(\sigma(\theta_2)) + (\sigma^2(\theta_2) - \sigma(\theta_2))v^\Delta(\sigma(\theta_1)) \\ &= u(\sigma(\theta_2)) + (\sigma^2(\theta_2) - \sigma(\theta_2))v^\Delta(\sigma(\theta_1)) \\ &> u(\sigma(\theta_2)) + (\sigma^2(\theta_2) - \sigma(\theta_2))u^\Delta(\sigma(\theta_1)) = u(\sigma^2(\theta_2)), \end{aligned} \tag{4}$$

which gives a contradiction. Thus, we have $v^\Delta(\theta_1) \le u^\Delta(\theta_1)$ and $v^\Delta(\sigma(\theta_2)) \ge u^\Delta(\sigma(\theta_2))$.

It follows from Lemma 2.1, $v^\Delta(\theta_1) \le u^\Delta(\theta_1)$ and $v^\Delta(\sigma(\theta_2)) \ge u^\Delta(\sigma(\theta_2))$ that

$$\begin{aligned} 0 &\le r(\sigma(\theta_2))u(\sigma^2(\theta_2))[v^\Delta(\sigma(\theta_2)) - u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)u(\theta_1)[v^\Delta(\theta_1) - u^\Delta(\theta_1)] \le 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Repeating the same argument, we have a $t_2 \in (t_1, \sigma^2(\theta_2))$ such that $u(t_2) = v(t_2) > 0$.

Setting $t_3 := \rho(t_2)$. It is clear that

$$t_3 = \rho(t_2) = sup\{s \in \mathbf{T} | s < t_2\} \ge t_1$$

and

$\sigma(t_3) = \sigma(\rho(t_2)) = t_2 < \sigma^2(\theta_2)$, (cf :page-2 and 3 of Bohner and Peterson [3]),

which imply $t_1 \le t_3 < \sigma(\theta_2)$.

Now, we separate the proof into the following cases:

Case (2.2a). Assume that $t_1 = t_3 = \rho(t_2)$. Since $t_1 < t_2 = \sigma(\rho(t_2)) = \sigma(t_1)$, t_1 is right-scattered. By Theorem 1.16 of Bohner and Peterson [3], we obtain

$$v^{\Delta}(t_1) = \frac{v(\sigma(t_1)) - v(t_1)}{\sigma(t_1) - t_1} = \frac{u(\sigma(t_1)) - u(t_1)}{\sigma(t_1) - t_1} = u^{\Delta}(t_1).$$

Follows from

$$v(t_1) = y(t_1) > 0, \quad v^{\Delta}(t_1) = u^{\Delta}(t_1)$$

and the property of uniqueness of positive solutions of (E) for non-zero initial condition, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

Case (2.2b). Assume that $t_1 < t_3 = \rho(t_2)$. Now, we claim that u and v intersect in $(t_1, t_2) = (t_1, \sigma(t_3))$. Assume, on the contrary, that $u(t) > v(t)$ in $(t_1, t_2) = (t_1, \sigma(t_3))$. From Lemma 2.1, $u^{\Delta}(t_1) \geq v^{\Delta}(t_1)$ and $u^{\Delta}(t_3) \leq v^{\Delta}(t_3)$ (cf: (3) and (4)),we see that

$$\begin{aligned} 0 &\leq r(t_3)u(\sigma(t_3))[v^{\Delta}(t_3) - u^{\Delta}(t_3)] = w(t_3) \\ &< w(t_1) = r(t_1)u(t_1)[v^{\Delta}(t_1) - u^{\Delta}(t_1)] \leq 0, \end{aligned}$$

which gives a contradiction, too.

Hence, there exists a $t_4 \in (t_1, t_3)$ such that $u(t_4) = v(t_4) > 0$.

Repeating the same argument, we have either

$$u(t) = v(t) \quad \text{on} \quad [\theta_1, \sigma^2(\theta_2)]$$

or a strictly decreasing sequence $\{t_n\}_{n=3}^{\infty} \subset [t_1, t_3] \subset (\theta_1, \sigma(\theta_2)]$ such that $t_n \in (t_1, t_{n-1})$, $u(t_{2n}) = v(t_{2n}) > 0$ and $\sigma(t_{2n+1}) = t_{2n}$ for all $n = 2, 3, \ldots$. Let

$$\eta := lim_{n\to\infty} t_n = lim_{n\to\infty} t_{2n} = lim_{n\to\infty} \sigma(t_{2n+1}) = \sigma(\eta) \in [t_1, t_3] \subseteq (\theta_1, \sigma(\theta_2)].$$

Thus, η is right-dense, which implies $u(\eta) = v(\eta) > 0$ and

$$\begin{aligned} u^{\Delta}(\eta) &= lim_{n\to\infty} \tfrac{u(t_{2n+2}) - u(t_{2n})}{t_{2n+2} - t_{2n}} \\ &= lim_{n\to\infty} \tfrac{v(t_{2n+2}) - v(t_{2n})}{t_{2n+2} - t_{2n}} = v^{\Delta}(\eta). \end{aligned}$$

Therefore, by the property of uniqueness of positive solutions of (E) for non-zero initial condition, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction. smallskip Thus, we obtain the desired results.

Theorem 2.3. Suppose that there is no $t \in (\theta_1, \sigma^2(\theta_2))$ such that t is left-dense and right-scattered, simultaneously.. Then, the boundary value problem (E)-(BC.2) has at most one positive solution on $[\theta_1, \sigma^2(\theta_2)]$.
Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)-(BC.2) on $[\theta_1, \sigma^2(\theta_2)]$. Just as in the proof of Theorem 2.2, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$.

It follows from Lemma 2.1, $v^{\Delta}(\theta_1) \leq u^{\Delta}(\theta_1)$ and $v(\sigma(\theta_2)) \leq u(\sigma(\theta_2))$ (cf: (3) and (4))that

$$\begin{aligned} 0 &\leq r(\sigma(\theta_2))u^{\Delta}(\sigma(\theta_2))[u(\sigma(\theta_2)) - v(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)u(\theta_1)[v^{\Delta}(\theta_1) - u^{\Delta}(\theta_1)] \leq 0, \end{aligned}$$

which gives a contradiction. Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Since

$$u(\theta_1) = v(\theta_1) \quad \text{and} \quad u(t_1) = v(t_1) > 0,$$

repeating the same argument in proof of Theorem 2.2 that

$$u(t) = v(t) \quad \text{on} \quad [\theta_1, \sigma^2(\theta_2)]$$

. This gives a contradiction. Thus, we obtain the desired results.

Theorem 2.4. Suppose that there is no $t \in (\theta_1, \sigma^2(\theta_2))$ such that t is left-dense and right-scattered, simultaneously.. Then, the boundary value problem (E)-(BC.3) has at most one positive solution on $[\theta_1, \sigma^2(\theta_2)]$.
Proof. Assume to the contrary that u and v are two distinct positive solutions of (E)-(BC.3)on $[\theta_1, \sigma^2(\theta_2)]$. We claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1, $u^{\Delta}(\theta_1) = v^{\Delta}(\theta_1) = \xi_1 \leq 0$, $v(\theta_1) \leq u(\theta_1)$ and $v^{\Delta}(\sigma(\theta_2)) \geq u^{\Delta}(\sigma(\theta_2))$ (cf: (3) and (4))that

$$\begin{aligned} 0 &\leq r(\sigma(\theta_2))u(\sigma^2(\theta_2))[v^{\Delta}(\sigma(\theta_2)) - u^{\Delta}(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)u^{\Delta}(\theta_1)[u(\theta_1) - v(\theta_1)] \leq 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Follows from

$$u(t_1) = v(t_1) > 0 \quad \text{and} \quad u(\sigma^2(\theta_2)) = u(\sigma^2(\theta_2))$$

and Theorem2.2 that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction. Thus, we obtain the desired results.

Theorem 2.5 (Main result). Suppose that there is no $t \in (\theta_1, \sigma^2(\theta_2))$ such that t is left-dense and right-scattered, simultaneously.. Then, the boundary value problem (BVP) has at most one positive solution on $[\theta_1, \sigma^2(\theta_2)]$.

Proof. Assume to the contrary that u and v are two distinct positive solutions of (BVP) on $[\theta_1, \sigma^2(\theta_2)]$. We separate the proof into the following cases: Case (2.5a). Assume that $\alpha_1 = 0$, that is $u^\Delta(\theta_1) = v^\Delta(\theta_1) = 0$. Since $r(t)u^\Delta(t)$ and $r(t)v^\Delta(t)$ are strictly decreasing in $(\theta_1, \sigma(\theta_2))$, $u^\Delta(t) < 0$ and $v^\Delta(t) < 0$ on $(\theta_1, \sigma(\theta_2)]$.

(1^0). If $\alpha_2 = 0$, then $u^\Delta(\sigma(\theta_2)) = v^\Delta(\sigma(\theta_2)) = 0$. This contradicts to the fact that $u^\Delta(t) < 0$ and $v^\Delta(t) < 0$ on $(\theta_1, \sigma(\theta_2)]$.

(2^0). If $\beta_2 = 0$, then $u(\sigma^2(\theta_2)) = v(\sigma^2(\theta_2)) = 0$. It follows from Theorem 2.4 that $u(t) = v(t)$ on $[\theta_1, \theta_2]$, which gives a contradiction.

(3^0). If $\alpha_2\beta_2 \neq 0$, then

$$u^\Delta(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^\Delta(\sigma(\theta_2))u(\sigma^2(\theta_2)).$$

Now, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1, $u^\Delta(\theta_1) = v^\Delta(\theta_1) = 0$ and $u^\Delta(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^\Delta(\sigma(\theta_2))u(\sigma^2(\theta_2))$ that

$$\begin{aligned} 0 &= r(\sigma(\theta_2))[u(\sigma^2(\theta_2))v^\Delta(\sigma(\theta_2)) - v(\sigma^2(\theta_2))u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^\Delta(\theta_1) - v(\theta_1)u^\Delta(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$.
Setting $t_2 := \rho(t_1)$. It is clear that

$$t_2 = \rho(t_1) = sup\{s \in \mathbf{T} | s < t_1\} \geq \theta_1$$

and

$$\sigma(t_2) = \sigma(\rho(t_1)) = t_1 < \sigma^2(\theta_2), \quad (\text{ cf :page-3 of Bohner and Peterson [3]}),$$

which imply $\theta_1 \leq t_2 < \sigma(\theta_2)$.

Now, we separate the proof into the following sub-cases:

Sub-Case (2.5a-1). Assume that $\theta_1 = t_2 = \rho(t_1)$. Since $t_2 = \theta_1 < t_1 = \sigma(\rho(t_1)) = \sigma(t_2)$, t_2 is right-scattered. By Theorem 1.16 of Bohner and Peterson [3], we obtain

$$v^{\Delta}(t_2) = \frac{v(\sigma(t_2)) - v(t_2)}{\sigma(t_2) - t_2} = \frac{u(\sigma(t_2)) - u(t_2)}{\sigma(t_2) - t_2} = u^{\Delta}(t_2) = u^{\Delta}(\theta_1) = u^{\Delta}(\theta_1) = 0.$$

Follows from

$$v(t_2) = y(t_2) = v(\theta_1) = u(\theta_1) = v(t_1) = u(t_1) > 0, \quad v^{\Delta}(t_2) = u^{\Delta}(t_2)$$

and the property of uniqueness of positive solutions of (E) for non-zero initial condition, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.
Sub-Case (2.5a-2). Assume that $\theta_1 < t_2 = \rho(t_1)$. Now, we claim that u and v intersect in $(\theta_1, t_1) = (\theta_1, \sigma(t_2))$. Assume, on the contrary, that $u(t) > v(t)$ in $(\theta_1, t_1) = (\theta_1, \sigma(t_2))$. From Lemma 2.1, $u^{\Delta}(\theta_1) = v^{\Delta}(\theta_1) = 0$ and $u^{\Delta}(t_2) \le v^{\Delta}(t_2)$ (cf: (3) and (4)),we see that

$$\begin{aligned} 0 &\le r(t_2)u(\sigma(t_2))[v^{\Delta}(t_2) - u^{\Delta}(t_2)] = w(t_2) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^{\Delta}(\theta_1) - u(\theta_1)u^{\Delta}(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction, too.

Hence, there exists a $t_4 \in (t_1, t_3)$ such that $u(t_4) = v(t_4) > 0$. Since

$$u(t_1) = v(t_1) \quad \text{and} \quad u(t_4) = v(t_4) > 0,$$

repeating the same argument in proof of Theorem 2.2 that

$$u(t) = v(t) \quad \text{on} \quad [\theta_1, \sigma^2(\theta_2)]$$

. This gives a contradiction.

Case (2.5b). Assume that $\beta_1 = 0$, that is $u(\theta_1) = v(\theta_1) = 0$.

(4^0). If $\alpha_2 = 0$, then $u^{\Delta}(\sigma(\theta_2)) = v^{\Delta}(\sigma(\theta_2)) = 0$. It follows from Theorem 2.3 that $u(t) = v(t)$ on $[\theta_1, \theta_2]$, which gives a contradiction.

(5^0). If $\beta_2 = 0$, then $u(\sigma^2(\theta_2)) = v(\sigma^2(\theta_2)) = 0$. It follows from Theorem 2.2 that $u(t) = v(t)$ on $[\theta_1, \theta_2]$, which gives a contradiction.

(6^0). If $\alpha_2\beta_2 \neq 0$, then

$$u^{\Delta}(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^{\Delta}(\sigma(\theta_2))u(\sigma^2(\theta_2)).$$

Now, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1, $u(\theta_1) = v(\theta_1) = 0$ and $u^\Delta(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^\Delta(\sigma(\theta_2))u(\sigma^2(\theta_2))$ that

$$\begin{aligned} 0 &= r(\sigma(\theta_2))[u(\sigma^2(\theta_2))v^\Delta(\sigma(\theta_2)) - v(\sigma^2(\theta_2))u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^\Delta(\theta_1) - v(\theta_1)u^\Delta(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Since

$$u(\theta_1) = v(\theta_1) \quad \text{and} \quad u(t_1) = v(t_1) > 0,$$

repeating the same argument in proof of Theorem 2.2 that

$$u(t) = v(t) \quad \text{on} \quad [\theta_1, \sigma^2(\theta_2)]$$

. This gives a contradiction.

Case (2.5c). Assume that $\alpha_2 = 0$, that is $u^\Delta(\sigma(\theta_2)) = v^\Delta(\sigma(\theta_2)) = 0$. Since $r(t)u^\Delta(t)$ and $r(t)v^\Delta(t)$ are strictly decreasing in $(\theta_1, \sigma(\theta_2))$, $u^\Delta(t) > 0$ and $v^\Delta(t) > 0$ on $[\theta_1, \sigma(\theta_2))$.

(7^0). If $\alpha_1 = 0$, then $u^\Delta(\theta_1) = v^\Delta(\theta_1) = 0$ which contradicts to the fact $u^\Delta(t) > 0$ and $v^\Delta(t) > 0$ on $[\theta_1, \sigma(\theta_2))$.

(8^0). If $\beta_1 = 0$, then $u(\theta_1) = v(\theta_1) = 0$. By Theorem 2.3, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

(9^0). If $\alpha_1\beta_1 \neq 0$, then

$$u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1).$$

Now, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1, $u^\Delta(\sigma(\theta_2)) = v^\Delta(\sigma(\theta_2)) = 0$ and $u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1)$ that

$$\begin{aligned} 0 &= r(\sigma(\theta_2))[u(\sigma^2(\theta_2))v^\Delta(\sigma(\theta_2)) - v(\sigma^2(\theta_2))u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^\Delta(\theta_1) - v(\theta_1)u^\Delta(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Since

$$u(t_1) = v(t_1) > 0 \quad \text{and} \quad u^\Delta(\sigma(\theta_2)) = u^\Delta(\sigma(\theta_2)),$$

it follows from Theorem 2.3 that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

Case (2.5d). Assume that $\beta_2 = 0$, that is $u(\sigma^2(\theta_2)) = v(\sigma^2(\theta_2)) = 0$.

(10^0). If $\alpha_1 = 0$, then $u^\Delta(\theta_1) = v^\Delta(\theta_1) = 0$. By Theorem 2.4, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

(11^0). If $\beta_1 = 0$, then $u(\theta_1) = v(\theta_1) = 0$. By Theorem 2.2, we see that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

(12^0). If $\alpha_1\beta_1 \neq 0$, then

$$u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1).$$

Now, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1, $u(\sigma^2(\theta_2)) = v(\sigma^2(\theta_2)) = 0$ and $u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1)$ that

$$\begin{aligned} 0 &= r(\sigma(\theta_2))[u(\sigma^2(\theta_2))v^\Delta(\sigma(\theta_2)) - v(\sigma^2(\theta_2))u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^\Delta(\theta_1) - v(\theta_1)u^\Delta(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Since

$$u(t_1) = v(t_1) > 0 \quad \text{and} \quad u(\sigma^2(\theta_2)) = v(\sigma^2(\theta_2)),$$

it follows from Theorem 2.2 that $u(t) = v(t)$ on $[\theta_1, \sigma^2(\theta_2)]$. This gives a contradiction.

Case (2.5e). Assume that $\alpha_1\alpha_2\beta_1\beta_2 \neq 0$, that is

$$u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1)$$

and

$$u^\Delta(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^\Delta(\sigma(\theta_2))u(\sigma^2(\theta_2)).$$

Now, we claim that u and v intersect in $(\theta_1, \sigma^2(\theta_2))$. Suppose, on the contrary, that $u(t) > v(t)$ in $(\theta_1, \sigma^2(\theta_2))$. It follows from Lemma 2.1,

$$u^\Delta(\theta_1)v(\theta_1) = v^\Delta(\theta_1)u(\theta_1)$$

and

$$u^\Delta(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^\Delta(\sigma(\theta_2))u(\sigma^2(\theta_2))$$

that

$$\begin{aligned} 0 &= r(\sigma(\theta_2))[u(\sigma^2(\theta_2))v^\Delta(\sigma(\theta_2)) - v(\sigma^2(\theta_2))u^\Delta(\sigma(\theta_2))] = w(\sigma(\theta_2)) \\ &< w(\theta_1) = r(\theta_1)[u(\theta_1)v^\Delta(\theta_1) - v(\theta_1)u^\Delta(\theta_1)] = 0, \end{aligned}$$

which gives a contradiction.

Hence, there exists a $t_1 \in (\theta_1, \sigma^2(\theta_2))$ such that $u(t_1) = v(t_1) > 0$. Since $u(t_1) = v(t_1) > 0$ and

$$u^{\Delta}(\sigma(\theta_2))v(\sigma^2(\theta_2)) = v^{\Delta}(\sigma(\theta_2))u(\sigma^2(\theta_2)),$$

repeating the same argument in proof Case (2.5b)-(6^0) that

$$u(t) = v(t) \text{ on } [\theta_1, \sigma^2(\theta_2)].$$

This gives a contradiction.

By Cases (2.5a)-(2.5e), we obtain the desired result.

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