International Journal of Differential Equations and Applications

**Volume 10 No. 3 2005, 321-339**

# RELATION OF POSITIVE SOLUTIONS FOR SUPERLINEAR M-LAPLACIAN BVPS AND ITS APPLICATIONS

Fu-Hsiang Wong[1], Chien-Hsiung Lin[2], Wei-Cheng Lian[3]

[1] Department of Mathematics
National Taipei University of Education
134, Ho-Ping E. Rd, Sec. 2
Taipei 10659, TAIWAN
e-mail: wong@tea.ntue.edu.tw

[2]Department of Hospitality Marketing Management
National Kaohsiung Hospitality College
Kaohsiung 812, TAIWAN
e-mail: Chlin999@ms3.hinet.net

[3]Department of Information Management
National Kaohsiung Marine University
No. 142, Hai Chuan Road, Nan-Tzu Dist.
Kaohsiung 811, Taiwan
e-mail: wclian@mail.nkmu.edu.tw

**Abstract:** In this article, we will classify the relations of two distinct positive solutions for the superlinear m-Laplacian boundary value problems

$$(BVP) \qquad \begin{cases} (E) \;\; (\psi(r)|u'|^{m-2}u')' + \psi(r)f(r,u) = 0 \qquad \text{in} \qquad (0,1) \\ (BC) \;\; u'(0) = u(1) = 0. \end{cases}$$

Furthermore, we will use these relations of positive solutions to establish some criterion for the uniqueness of positive solutions of (BVP).

**AMS Subject Classification**: 34B15, 47H15
**Key Words**: boundary value problems, positive solution, uniqueness, superlinear

Received: October 30, 2006 © 2005 Academic Publications

## 1. Introduction

In this paper, we are concerned the relations of two distinct positive solutions of the following m-Laplacian boundary value problems

$$(BVP) \quad \begin{cases} (E)(\psi(r)|u'|^{m-2}u')' + \psi(r)f(r,u) = 0 \text{ in } (0,1), \\ (BC) \ \ u'(0) = u(1) = 0, \end{cases}$$

where $m \geq 2$.

These problems arise, for example, in the study of the uniqueness and existence of positive solutions of boundary value problems for m-Laplacian nonlinear differential equation

$$(E_1) \qquad (|u'|^{m-2}u')' + f(r,u) = 0, \qquad \text{in} \quad (0,1)$$

subject to one of the following sets of boundary conditions:

$$(BC.1) \qquad u(0) = \xi_1 \geq 0, \quad u'(1) = \xi_2 \geq 0,$$

$$(BC.2) \qquad u'(0) = \xi_1 \leq 0, \quad u(1) = \xi_2 \geq 0,$$

$$(BC.3) \qquad u(0) = \xi_1 \geq 0, \quad u(1) = \xi_2 \geq 0,$$

and

$$(BC^*) \quad \begin{cases} \alpha_1 u(0) - \beta_1 u'(1) = 0 \\ \alpha_2 u(0) + \beta_2 u'(1) = 0. \end{cases}$$

In 1998, Wong [18] show that

**Theorem 1.** *(Relation- Wong [18]) Suppose that $m \geq 2$ and $f : (0,1) \times (0,\infty) \to (0,\infty)$ satisfy*

$(H_1)$ *$f(r,x)$ is locally Lipschitz continuous for $x \in (0,\infty)$ and $f(r,x)/x^{m-1}$ is strictly decreasing with respect to $x \in (0,\infty)$ for each fixed $r \in (0,1)$.*

*Let $u$, $v$ be two distinct positive solutions of $(E_1) - (BC_i)$ for $i = 1,\ 2,\ 3$ and define*

$$w(r) := \{u(r)\}^{m-1}(|v'(r)|^{m-2}v'(r)) - (|u'(r)|^{m-2}u'(r))\{v(r)\}^{m-1}$$

*for $r \in [0,1]$. Then, $w'(r) < 0$ in $(0,1)$.*

Furthermore, Wong [18] use this relation to establish the following uniqueness theorems:

**Theorem 2.** *(Uniqueness theorem - Wong [18]) Suppose that $m \geq 2$ and $f(r,x)$ satisfy the hypothesis $(H_1)$. Then the boundary value problems $(E_1)-(BC.i)$ and $(E_1)-(BC^*)$ for $i = 1,\ 2,\ 3,$ have at most one positive solution in $C^1([0,1])$.*

In 1999, Wong [20] use the uniqueness theorem 2 to obtain the following existence theorem:

**Theorem 3.** *(Existence theorem - Wong [20]) Suppose that $m \geq 2$ and $f : [0,1) \times (0,\infty) \to (0,\infty)$ satisfy the hypotheses $(H_1)$ and*

*$(H_2)$* $\quad \sup\limits_{r\in[0,1)} \lim\limits_{\alpha\to\infty} \frac{f(r,\alpha)}{\alpha^{m-1}} = 0,$

*$(H_3)$* $\quad \inf\limits_{r\in[0,1)} \lim\limits_{\alpha\to 0^+} \frac{f(r,\alpha)}{\alpha^{m-1}} = \infty.$

*Then the boundary value problem*

$$(BVP_1) \qquad \begin{cases} (E_1) \quad (|u'(r)|^{m-2}u'(r))' + f(r,u(r)) = 0, \quad \text{in} \quad (0,1), \\ (BC) \quad u'(0) = u(1) = 0, \end{cases}$$

*has a positive solution in $C^2[0,1) \cap C[0,1]$.*

It is clear that the main assumption of the above-mentioned results is $f(r,x)/x^{m-1}$ **is strictly decreasing with respect to** $x \in (0,\infty)$

Inspire by the this observation, we can not but ask **"whether or not we can construct a similar result** for $f$ satisfying $f(r,x)/x^{m-1}$ **is strictly increasing with respect to** $x \in (0,\infty)$.

There has been considerable interest in the uniqueness and existence of positive solutions of

$$(BVP_2) \qquad \begin{cases} (E_2) \quad (r^{n-1}|u'|^{m-2}u')' + r^{n-1}(u^p + \lambda u) = 0 \quad \text{in} \quad (0,1) \\ (BC) \quad u'(0) = u(1) = 0, \end{cases}$$

where $n \geq 1$, $m \geq 2$, $1 < p < \infty$ and $\lambda \geq 0$.

For $\lambda = 0$, see, for example, Gidas, Ni and Nirenberg [7]. For $\lambda = 0$ and $1 < p \leq \frac{n}{n-2}$, see Ni and Nussbaum [14]; their result has been extended to $1 < p < \frac{n+2}{n-2}$ by Kwong and Li [11]. Later, Ligun [13] and Srikanth [16] have established the uniqueness of solutions for $\lambda \geq 0$ and $1 < p \leq \frac{n+2}{n-2}$. For the related results, we refer to Agarwal and Lakshmikantham [2],Agarwal, Wong and Yu[3], Brezis and Oswald [5], Dalmasso [6], Knaap and Peletier [8], Kwong [9, 10], Kwong and Li[11], Lian and Wong [12], Naito [15], Wong and Yu [17], Wong [18] and the references therein.

## 2. Main Result

Recently, Adimurthi and Yadava [1], Ambrosetti, Azorero and Peral [4] applied a Pohožaev-type variational identity and Schauder's fixed point theorem to establish the following uniqueness and non-uniqueness theorems:

**Theorem 4.** *(Uniqueness - Adimurthi and Yadava [1]) Suppose that* $\lambda \geq 0$, $m = q+1 < p+1 \leq \frac{mn}{n-m}$ *and* $2 \leq m < n$.

*Then, the boundary value problem*

$$(BVP_3) \qquad \begin{cases} (E_3) \ \ (r^{n-1}|u'|^{m-2}u')' + (u^p + \lambda u^q)r^{n-1} = 0 \qquad \mathit{in} \qquad (0,1) \\ (BC) \ \ u'(0) = u(1) = 0, \end{cases}$$

*has at most one positive solution.*

**Theorem 5.** *(Non-uniqueness - Ambrosetti, Azorero and Peral [4]) Suppose that* $\lambda \geq 0$, $2 \leq q+1 < m < p+1 \leq \frac{mn}{n-m}$ *and* $2 \leq m < n$.

Then, the boundary value problem $(BVP_3)$ has at least two positive solutions.

Moreover, if $\lambda \geq 0$, $\quad 2 \leq m < n$ and $m < \frac{mn}{n-m} \leq p+1 \leq q+1$, then we have

$$\begin{aligned} 0 \geq \ & \left\{\frac{n}{p+1} - \frac{n-m}{m}\right\} \int_0^1 r^{n-1}|u'|^m dr + \lambda n \left\{\frac{1}{q+1} - \frac{1}{p+1}\right\} \int_0^1 r^{n-1}u^{q+1} dr \\ & = (\frac{m-1}{m})|u'(1)|^m + \frac{n}{p+1} u(1)|u'(1)|^{m-2}u'(1) + \frac{u^{p+1}(1)}{p+1} + \frac{\lambda u^{q+1}(1)}{q+1} > 0. \end{aligned}$$

This contradiction implies $(BVP_3)$ has no positive solutions.

There is another excellent and interesting result which is due to Dalmasso [6].

**Theorem 6.** *(Uniqueness - Dalmasso [6]) Let* $f \in C^1([0,R] \times [0,\infty))$ *satisfying*

$(H_4)$ $uf_u(r,u) > f(r,u)$ *for* $(r,u) \in [0,R] \times [0,\infty)$, *that is,* $\frac{f(r,u)}{u}$ *is strictly increasing in* $u \in [0,\infty)$ *for each fixed* $r \in [0,R]$,

$(H_5)$ $f_r(r,u) \le 0$ *for* $(r,u) \in [0,R] \times [0,\infty)$,

$(H_6)$ *there exists* $u > 0$ *such that* $f(R,u) \ge 0$.

*Then,*

$$(BVP_4) \qquad \begin{cases} (E_4) \quad u''(r) + f(|r|, u(r)) = 0, & -R < r < R, \\ (BC_4) \quad u(-R) = u(R) = 0. \end{cases}$$

*has at most one positive solution in* $C^2[-R,R]$.

The purpose of the paper is to classify the relations of two distinct positie solutions and use these relations to establish a simple criterion for the uniqueness of positive solutions of $(BVP)$.

**Remark 7.** Investigate the proof of Adimurthi and Yadava [1], and Ambrosetti, Azorero and Peral [4], we see that the uniqueness and non-uniqueness results appear in [1] rely on the following two well-known Lemmas:

$(L_1)$ the special function

$$f(r,u) := u^p + \lambda u^q \ \ (\text{ for } \quad \lambda \ge 0 \quad \text{and} \quad q+1 \le m),$$

which guarantee the uniqueness of positive solution of $(E)$ with respect to **non-negative initial condition**, i.e : if $u$ and $v$ are two solutions of $(E)$ satisfy

$$(IC^*) \ \ u(r_0) = v(r_0) > 0 \quad \text{and} \quad u'(r_0) = v'(r_0) \le 0 \quad \text{for} \quad r_0 \in [0,1],$$

then $u(r) \equiv v(r)$ on $[0,1]$.

$(L_2)$ the special function

$$f(r,u) := u^p + \lambda u^q \ \ (\text{ for } \quad \lambda \ge 0 \quad \text{and} \quad q+1 \le m),$$

which guarantee the uniqueness of positive solution of $(BVP)$ with respect to **monotonic property**, i.e : if $u$ and $v$ are two solutions of $(BVP)$ satisfy $v(r) \le u(r)$ on $[0,1]$, then $u(r) \equiv v(r)$ on $[0,1]$.

Here, we shall point out there are **a few** functions satisfy $(L_1)$ and $(L_2)$. For example, consider

$$((|u'|^2u')' + 24(1-u) = 0; \qquad (IC^{**}) \ \ u(0) = 1, \quad u'(1) = 0,$$

the forcing term is infinitely smooth. Nevertheless, there are at least three solutions:

$$u_1(r) := 1, \quad u_2(r) := 1 - r^2 \quad \text{and} \quad u_3(r) := 1 + r^3.$$

Throughout this section, we assume that the following conditions hold:

$(C_1)$ $f \in C([0,1] \times (0,\infty) \times (-\infty,\infty); (0,\infty))$ and $\psi : [0,1] \longrightarrow [0,\infty)$ such that $\psi(r) > 0$, $\psi'(r) \geq 0$ on $(0,\infty)$,

$(C_2)$ $f(r,x)/x^{m-1}$ is strictly increasing with respect to $x \in (0,\infty)$ for each fixed $r \in (0,1)$.

$(C_3)$ $f(r,x)$ guarantee the uniqueness of positive solution of equation $(E)$ with respect to the **non-zero initial condition** i.e : if $u$ and $v$ are two solutions of $(E)$ satisfy

$$(IC) \quad u(r_0) = v(r_0) > 0 \quad \text{and} \quad u'(r_0) = v'(r_0) < 0 \quad \text{for} \quad r_0 \in (0,1),$$

then $u(r) \equiv v(r)$ on $[0,1]$.

**Remark 8.** Here, we shall remark that

$(N_1)$ the assumption $(C_2)$ implies that for each $k \in \{0, 1, \cdots, m-1\}$, $\frac{f(r,x)}{x^k}$ is strictly increasing with respect to $u \in (0,\infty)$ for each fixed $r \in (0,1)$;

$(N_2)$ there are many functions guarantee the uniqueness of positive solution of equation $(E)$ with respect to nonzero initial condition.

For example, $f(r,x)$ is locally Lipschitz continuous with respect to $x \in (0,\infty)$ for each fixed $r \in (0,1)$.

$(N_3)$ it follows from $(C_1)$ that we have that $u \in C^2(0,1) \cap C^1[0,1]$ and $u' < 0$ in (0,1]. Therefore, we can rewrite equation $(E)$ as

$$(E_5) \qquad (\psi(r)(-u')^{m-2}u')' + \psi(r)f(r,u) = 0 \qquad \text{in} \qquad (0,1),$$

or

$$(E_6) \qquad (m-1)u'' + \frac{\psi'(r)}{\psi(r)}u' + |u'|^{2-m}f(r,u) = 0 \qquad \text{in} \qquad (0,1).$$

In order to treat our main result, we need the following lemmas:

**Lemma 9.** *(Adimurthi and Yadava [1, Lemma 2.2]) Let $u$ and $v$ be two nonnegative $C^1$ functions on $[R_1, R_2]$, with $v \not\equiv 0$ and $u > 0$ in $(R_1, R_2)$. Assume that $v(R_1) = 0$ and $u'(R_1) \neq 0$ if $u(R_1) = 0$, and that $v(R_2) = 0$ and*

*$u'(R_2) \neq 0$ if $u(R_2) = 0$. Then, there exists a unique $t > 0$ and $\xi \in [R_1, R_2]$ such that $w(r) = u(r) - tv(r) \geq 0$ on $[R_1, R_2]$ and $w(\xi) = 0$. Furthermore,*

*(i) d If $\xi \in (R_1, R_2)$ and $w$ is in $C^2$ in a neighborhood of $\xi$, then $w(\xi) = w'(\xi) = 0$ and $w''(\xi) \geq 0$;*

*(ii) if $\xi = R_i$ and $u(R_i) = 0$ for some $i \in \{1, 2\}$, then $w'(\xi) = 0$;*

*(iii) if $u(R_1) \neq 0$ and $v(R_1) = 0$, then $\xi \neq R_1$;*

*(iv) if $u(R_2) \neq 0$ and $v(R_2) = 0$, then $\xi \neq R_2$.*

**Lemma 10.** *Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$ and $w(r) = u(r) - tv(r)$ on $[0, 1]$, where $t > 0$ is fixed. Then we have the following results:*

$(R_1)$ *If $w(r_1) = w'(r_1) = 0, w''(r_1) \geq 0$ for some $r_1 \in (0, 1)$, then $t < 1$;*

$(R_2)$ *If $w(0) = 0$ $(i.e., t = u(0)/v(0))$ and $w(r) > 0$ in $(0, r_2)$ for some $r_2 \in (0, 1)$, then $t < 1$.*

*Proof.* $(R_1)$ Suppose that $w(r_1) = w'(r_1) = 0$ and $w''(r_1) \geq 0$. Since $u$ and $v$ satisfy

$$(m-1)u'' + \frac{\psi'(r)}{\psi(r)}u' + |u'|^{2-m} f(r, u) = 0$$

and

$$(m-1)v'' + \frac{\psi'(r)}{\psi(r)}v' + |v'|^{2-m} f(r, v) = 0,$$

we have

$$\begin{aligned} 0 &\geq -[(m-1)w''(r_1) + \frac{\psi'(r)}{\psi(r)}w'(r_1)] \\ &= |u'(r_1)|^{2-m}[f(r_1, u(r_1)) - t^{m-1} f(r_1, v(r_1))]. \end{aligned}$$

Then, we obtain that

$$v^{m-1}(r_1) f(r_1, u(r_1)) \leq (tv(r_1))^{m-1} f(r_1, v(r_1)) = u(r_1)^{m-1} f(r_1, v(r_1)),$$

which implies

(1) $$\frac{f(r_1, u(r_1))}{u^{m-1}(r_1)} \leq \frac{f(r_1, v(r_1))}{v^{m-1}(r_1)}.$$

It is clear that $t \neq 1$. In fact, if $t = 1$, it follows from $(C_3)$ that $u \equiv v$ on $[0, 1]$ (by using $u(r_1) = v(r_1) > 0$ and $u'(r_1) = v'(r_1) < 0$). This gives a contradiction.

Thus, we may complete the proof of $(R_1)$ if we can exclude the possibility of $t > 1$. Assume that $t > 1$, it follows from $w(r_1) = 0$ that $u(r_1) = tv(r_1) > v(r_1) > 0$. Follows from $f(r, u)/u^{m-1}$ is strictly increasing with respect to $u$ that

$$\frac{f(r_1, u(r_1))}{u^{m-1}(r_1)} > \frac{f(r_1, v(r_1))}{v^{m-1}(r_1)},$$

which contradicts with (1). Thus, we see that $t < 1$.

$(R_2)$ Suppose, to the contrary, that $t \geq 1$. First, we claim that there is a $\epsilon_0 \in (0, r_2)$ such that $w'(r) \geq 0$ on $[0, \epsilon_0]$. Assume, to the contrary, that there exists a decreasing sequence $\{\epsilon_k\} \subset (0, r_2)$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and $w'(\epsilon_k) < 0$. Since $w(0) = 0$ and $w(r) > 0$ in $(0, r_2)$, for every $\epsilon_k \in (0, r_2)$, there exists a $\xi_k \in (0, \epsilon_k)$ such that $w'(\xi_k) > 0$. Hence, there exists a decreasing sequence $\{\eta_k\} \subset (0, r_2)$ such that $\lim_{k\to\infty} \eta_k = 0, w'(\eta_k) = 0$ and $w''(\eta_k) \geq 0$. Thus

$$\begin{aligned} 0 &\geq -[(m-1)w''(\eta_k) + \frac{\psi'(\eta_k)}{\psi(\eta_k)} w'(\eta_k)] \\ &= |u'(\eta_k)|^{2-m}[f(\eta_k, u(\eta_k)) - t^{m-1} f(\eta_k, v(\eta_k))], \end{aligned}$$

which implies

$$v^{m-1}(\eta_k) f(\eta_k, u(\eta_k)) \leq (tv(\eta_k))^{m-1} f(\eta_k, v(\eta_k)) \leq u^{m-1}(\eta_k) f(\eta_k, v(\eta_k)).$$

Therefore,

$$\frac{f(\eta_k, u(\eta_k))}{u^{m-1}(\eta_k)} \leq \frac{f(\eta_k, v(\eta_k))}{v^{m-1}(\eta_k)}. \tag{2}$$

Since $w(\eta_k) > 0$, $u(\eta_k) > tv(\eta_k) \geq v(\eta_k) > 0$. It follows from $f(r, u)/u^{m-1}$ is strictly increasing with respect to $u$ that

$$\frac{f(\eta_k, u(\eta_k))}{u^{m-1}(\eta_k)} > \frac{f(\eta_k, v(\eta_k))}{v^{m-1}(\eta_k)},$$

which contradicts with (2). Therefore, we see that there exists a $\epsilon_0 \in (0, r_2)$ such that $w'(r) \geq 0$ on $[0, \epsilon_0]$.

It follows from $w'(0) = 0$ and $w'(r) \geq 0$ on $[0, \epsilon_0]$, we see that there exists a decreasing sequence $\{\theta_k\}$ such that $\lim_{k\to\infty} \theta_k = 0$ and $w''(\theta_k) \geq 0$. Since $w'(\theta_k) \geq 0$,

$$\frac{|u'(\theta_k)|}{|v'(\theta_k)|} \leq t \qquad \text{for each} \qquad k$$

and

$$\begin{aligned} 0 &\geq -[(m-1)w''(\theta_k) + \frac{\psi'(\theta_k)}{\psi(\theta_k)} w'(\theta_k)] \\ &= \frac{f(\theta_k, u(\theta_k))}{|u'(\theta_k)|^{m-2}} - \frac{tf(\theta_k, v(\theta_k))}{|v'(\theta_k)|^{m-2}} \\ &\geq |u'(\theta_k)|^{2-m}[f(\theta_k, u(\theta_k)) - t^{m-1} f(\theta_k, v(\theta_k))], \end{aligned}$$

which implies

$$v^{m-1}(\theta_k) f(\theta_k, u(\theta_k)) \leq (tv(\theta_k))^{m-1} f(\theta_k, v(\theta_k)) \leq u^{m-1}(\theta_k) f(\theta_k, v(\theta_k)).$$

Therefore,

$$\frac{f(\theta_k, u(\theta_k))}{u^{m-1}(\theta_k)} \leq \frac{f(\theta_k, v(\theta_k))}{v^{m-1}(\theta_k)}. \tag{3}$$

Since $w(\theta_k) > 0$, $u(\theta_k) > tv(\theta_k) \geq v(\theta_k) > 0$.

It follows from $f(r, u)/u^{m-1}$ is strictly increasing with respect to $u$ that

$$\frac{f(\theta_k, u(\theta_k))}{u^{m-1}(\theta_k)} > \frac{f(\theta_k, v(\theta_k))}{v^{m-1}(\theta_k)},$$

which contradicts with (3). Thus, we obtain that $t < 1$. □

**Lemma 11.** *Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$. Then, there exist $t \in (0, 1)$ and $\xi \in [0, 1]$ such that for $w(r) = u(r) - tv(r)$, one the following holds:*

(*a*) $\xi = 0$ *and* $w(0) = 0$,

(*b*) $\xi \in (0, 1), w \geq 0$ *on* $[0, \xi]$, $w(\xi) = w'(\xi) = 0$ *and* $w''(\xi) \geq 0$,

(*c*) $\xi = 1, w \geq 0, w' \leq 0$ *on* $[0, 1]$, $w'(1) = 0$.

*Proof.* From Lemma 9 and (BC), we can choose $t_1 > 0, \xi_1 \in [0, 1]$ such that $w_1 = u - t_1 v$ satisfies $w_1 \geq 0$ on $[0, 1]$ and $w_1(\xi_1) = w_1'(\xi_1) = 0$.

Case (1). Suppose that $\xi_1 \in (0, 1)$ . It follows from Lemma 9-(i) and Lemma 10-$(R_1)$, we see that $w_1''(\xi_1) \geq 0$ and $t_1 < 1$. Thus, conclusion $(b)$ holds.

Case (2). Suppose that $\xi_1 = 0$, which implies $w_1(0) = 0$.

$(1^0)$ Suppose that $w_1(r) > 0$ in $(0, r_2)$ for some $r_2 \in (0,1)$. It follows from Lemma 10-$(R_2)$ that $t_1 < 1$, thus conclusion $(a)$ holds.

$(2^0)$ Suppose that $w_1(r)$ is identically zero on $[0, r_2]$. Let $\xi_1 = r_2 \in (0,1)$. Since $w_1(r) = w_1'(r) = w_1''(r) = 0$ on $[0, r_2] = [0, \xi_1]$ and Lemma 10-$(R_1)$, we see that $t_1 < 1$. Thus, the conclusion $(a)$ holds.

$(3^0)$ Suppose that there exists a decreasing sequence $\{\eta_k\} \subset (0,1)$ such that

$$\lim_{k\to\infty} \eta_k = 0, \quad w_1(\eta_k) = w_1'(\eta_k) = 0 \quad \text{and} \quad w_1''(\eta_k) \geq 0.$$

It follows from Lemma 10-$(R_1)$, we see that $t_1 < 1$. Hence, conclusion $(a)$ holds.

Case (3). Suppose that $\xi_1 = 1$.

It follows from Lemma 9 that for every $\eta \in (0,1)$, we can choose $t_\eta > 0, \xi_\eta \in [0, \eta]$ such that $w_\eta = u - t_\eta v$ satisfies

$$w_\eta(r) \geq 0 \quad \text{on} \quad [0, \eta] \quad \text{and} \quad w_\eta(\xi_\eta) = 0.$$

Now, we assume that $(a)$ and $(b)$ do not hold. Under this assumption, we claim that the conclusion $(c)$ has to hold.

$(4^0)$ Suppose that there exists a $\eta \in (0,1)$ such that $\xi_\eta \in (0, \eta) \subseteq (0,1)$. It follows from Lemma 9-(i) and Lemma 10-$(R_1)$, we see that

$$w_\eta(\xi_\eta) = w_\eta'(\xi_\eta) = 0, \quad w_\eta''(\xi_\eta) \geq 0 \quad \text{and} \quad t_1 < 1.$$

This contradicts to the assumption that conclusion $(b)$ does not hold.

$(5^0)$ Suppose that there exists a $\eta \in (0,1)$ such that $\xi_\eta = 0$. Repeat the same argument as in Case (2), we can show that the conclusion $(a)$ holds. This contradicts to the assumption that conclusion $(a)$ does not hold.

$(6^0)$ Suppose that $\xi_\eta = \eta$ for every $\eta \in (0,1)$. It follows from $w_\eta(\eta) = 0$ that $t_\eta = u(\eta)/v(\eta)$. Hence, for every pair $0 \leq r \leq \eta \leq 1$, we deduce from $w_\eta(r) \geq 0$ on $[0, \eta]$ that

$$\frac{u(r)}{v(r)} \geq t_\eta = \frac{u(\eta)}{v(\eta)}.$$

Therefore, $(u/v)(r)$ is decreasing on $[0,1]$. This implies that $\frac{u(r)}{v(r)} \leq \frac{u'(r)}{v'(r)}$ on $[0,1]$ (by using $u',\ v' < 0$).

Now, suppose that there exists a $\eta_1 \in (0,1)$ such that $w_1'(\eta_1) > 0$, then $u'(\eta_1)/v'(\eta_1) < t_1$ (by using $u',\ v' < 0$). Let $r \in (\eta_1, 1)$, then

$$\frac{u(r)}{v(r)} \le \frac{u(\eta_1)}{v(\eta_1)} \le \frac{u'(\eta_1)}{v'(\eta_1)} < t_1,$$

and hence $w_1(r) < 0$, which is a contradiction. Hence $w_1'(r) \le 0$ on $[0,1]$.

Next, we will show that $t_1 < 1$. Suppose that $t_1 \ge 1$, and let

$$H(r) := \psi(r)[|u'(r)|^{m-2}u'(r) - t_1^{m-1}|v'(r)|^{m-2}v'(r)] \quad \text{on} \quad [0,1].$$

(i) Assume that $t_1 > 1$. Since $u(r) \ge t_1 v(t) > v(r)$ on $(0,1)$,

$$\frac{f(r,u(r))}{u^{m-1}(r)} \ge \frac{f(r,t_1 v(r))}{(t_1 v(r))^{m-1}} > \frac{f(r,v(r))}{v^{m-1}(r)} \quad \text{on} \quad (0,1).$$

This implies

$$f(r,u(r)) > \frac{u^{m-1}(r)}{v^{m-1}(r)} f(r,v(r)) \ge t_1^{m-1} f(r,v(r)) \quad \text{on} \quad [0,1),$$

thus we have

$$H'(r) = -\psi(r)[f(r,u(r)) - t_1^{m-1} f(r,v(r))] < 0 \quad \text{on} \quad (0,1).$$

It follows from $w'(0) = w'(1) = 0$ that $0 = H(1) < H(0) = 0$, which gives a contradiction.

(ii) Assume that $t_1 = 1$. Since $u(r) \ge t_1 v(t) = v(r)$ on $(0,1)$,

$$\frac{f(r,u(r))}{u^{m-1}(r)} \ge \frac{f(r,t_1 v(r))}{(t_1 v(r))^{m-1}} = \frac{f(r,v(r))}{v^{m-1}(r)} \quad \text{on} \quad (0,1).$$

This implies

$$f(r,u(r)) \ge \frac{u^{m-1}(r)}{v^{m-1}(r)} f(r,v(r)) \ge f(r,v(r)) \quad \text{on} \quad (0,1),$$

thus we have

$$H'(r) = -\psi(r)[f(r,u(r)) - f(r,v(r))] \le 0 \quad \text{on} \quad [0,1) \quad (\text{ by } \ (E)).$$

Since $u(r)$ and $v(r)$ are distinct, there exists a $\rho \in (0,1)$ such that $u(\rho) > v(\rho) > 0$. Repeat the same argument as above-mentioned, we see that $H'(\rho) < 0$. Follows from $w'(0) = w'(1) = 0$ and $H'(\rho) < 0$ that $0 = H(1) < H(\rho) < H(0) = 0$, which gives a contradiction.

By (i)-(ii), we see that $t_1 < 1$. Thus, conclusion $(c)$ holds.

By Cases (1)-(3), we complete the proof. □

**Lemma 12.** *Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$. Then, there exist $t \in (0,1)$ and $\xi \in [0,1]$ such that for $w(r) = u(r) - tv(r)$, one the following holds:*

$(a^*)$ $\xi = 0$ *and* $w(0) = 0$,

$(b^*)$ $\xi \in (0,1), w \geq 0, w' \leq 0$ *on* $[0,\xi]$, $w(\xi) = w'(\xi) = 0$,

$(c^*)$ $\xi = 1, w \geq 0, w' \leq 0$ *on* $[0,1]$, $w'(1) = 0$.

*Proof.* Follows from Lemma 11, we can see that it suffice to show conclusion $(b^*)$ has to hold, if conclusions $(a^*)$ and $(c^*)$ do not hold. Now, suppose that conclusions $(a^*)$ and $(c^*)$ do not hold, and consider the set

$$\mathbf{Z} := \left\{ \eta > 0 \quad \begin{array}{l} |\eta \ \text{ satisfies Lemma } 10-(b), \ \text{ i.e. there is a } t_\eta \in (0,1) \ \text{ such that} \\ |w_\eta := u - t_\eta v \geq 0 \ \text{ on } \ [0,\eta], \ \ w_\eta(\eta) = w'_\eta(\eta) = 0 \ \text{ and } \ w''_\eta(\eta) \geq 0 \end{array} \right\}$$

It follows from Lemma 11 that $\mathbf{Z}$ is non-empty, thus we see that

$$1 > \eta_0 := \inf \mathbf{Z} \geq 0.$$

Case (1). Suppose $\eta_0 = 0$. Then there exists a sequence $\eta_k \to 0$ with $t_{\eta_k} \to t_0 \leq 1$ such that $w_{\eta_k}(\eta_k) = 0$. This implies $u(0) = t_0 v(0)$. Repeat the same argument as in Lemma 11-Case (2), we can show that $t_0 < 1$. Thus, the conclusion $(a^*)$ holds. This contradicts to the assumption that conclusion $(a^*)$ does not hold.

Case (2). Suppose $\eta_0 > 0$. Now, by continuity, we have

$$w_{\eta_0}(\eta_0) = w'_{\eta_0}(\eta_0) = 0 \quad \text{ and } \quad w''_{\eta_0}(\eta_0) \geq 0.$$

From Lemma 10-$(R_1)$, we see that $t_{\eta_0} < 1$.

$(1^0)$ Suppose that there exists a $\eta \in (0,\eta_0)$ such that $\xi_\eta \in (0,\eta)$. It follows from Lemma 11-$(b)$, we see that

$$w_\eta(\xi_\eta) = w'_\eta(\xi_\eta) = 0, \quad w''_\eta(\xi_\eta) \geq 0 \quad \text{ and } \quad t_\eta < 1.$$

This implies $\eta_0 > \eta \geq \eta_0$, which gives a contradiction.

$(2^0)$ Suppose that there exists a $\eta \in (0,\eta_0)$ such that $\xi_\eta = 0$. Repeat the same argument as in Lemma 11-Case (2), we can show that the conclusion $(a^*)$ holds. This contradicts to the assumption that conclusion $(a^*)$ does not hold.

$(3^0)$ Suppose that $\xi_\eta = \eta$ for every $\eta \in (0, \eta_0)$. It follows from $w_\eta(\eta) = 0$ that $t_\eta = u(\eta)/v(\eta)$. Hence, for every pair $0 \le r \le \eta \le \eta_0$, we deduce from $w_\eta(r) \ge 0$ on $[0, \eta]$ that

$$\frac{u(r)}{v(r)} \ge t_\eta = \frac{u(\eta)}{v(\eta)}.$$

Therefore, $(u/v)(r)$ is decreasing on $[0, \eta_0]$. This implies that $\frac{u(r)}{v(r)} \le \frac{u'(r)}{v'(r)}$ on $[0, \eta_0]$ (by using $u',\ v' < 0$).

Now, suppose that there exists a $\eta_2 \in (0, \eta_0)$ such that $w'_{\eta_0}(\eta_2) > 0$, then $u'(\eta_2)/v'(\eta_2) < t_{\eta_0}$ (by using $u',\ v' < 0$). Let $r \in (\eta_2, \eta_0)$, then

$$\frac{u(r)}{v(r)} \le \frac{u(\eta_2)}{v(\eta_2)} \le \frac{u'(\eta_2)}{v'(\eta_2)} < t_{\eta_0},$$

and hence $w_{\eta_0}(r) < 0$, which is a contradiction. Hence $w'_{\eta_0} \le 0$ on $[0, \eta_0]$. Repeat the same argument as in Lemma 11-Case (3)-$(6^0)$, we see that $t_{\eta_0} < 1$. Thus, conclusion $(b^*)$ holds.

By Cases (1)-(2), we complete the proof. □

**Theorem 13.** *(Main Result) Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$. Then, there exist $t_1,\ t_2 \in (0, 1)$ and $\xi_1,\ \xi_2 \in [0, 1]$ such that one of $w_1(r) = u(r) - t_1 v(r)$ and $w_2(r) = v(r) - t_2 u(r)$, has to satisfy either $(b^*)$ or $(c^*)$ which appear in the Lemma 12.*

*Proof.* Suppose to the contrary that for both $w_1$ and $w_2$ satisfy $(a^*)$, then we have $t_1,\ t_2 \in (0, 1)$ and

$$\begin{cases} w_1(0) = u(0) - t_1 v(0) = 0, \\ w_2(0) = v(0) - t_2 u(0) = 0. \end{cases}$$

This implies that $t_1 t_2 = 1$, which contradicts to the fact that $t_1,\ t_2 \in (0, 1)$.

Therefore, one of $w_1$ and $w_2$ has to satisfy either $(b^*)$ or $(c^*)$. □

## 3. Applications for Uniqueness

**Theorem 14.** *(1-st Uniqueness theorem) Suppose that*

$(C_4)$ $\psi(r) = r^{n-1}$ *on* $[0,1]$, $f(r,u) := u^p + \lambda u^q$ *on* $[0,1]\times[0,\infty)$ *and satisfies*

$$2 \le m < n \quad \text{and} \quad m = q+1 < p+1 \le \frac{mn}{n-m}.$$

*Then for any* $\lambda \ge 0$, *boundary value problem (BVP) has at most one positive solution in* $C^2(0,1)\cap C^1[0,1]$.

*Proof.* Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$. Then it follows from Theorem 13 that there exist $t_i \in (0,1), \xi_i \in [0,1]$ for $i = 1,2$ such that each of the functions $w_1(r) = u(r) - t_1 v(r)$ and $w_2(r) = v(r) - t_2 u(r)$ satisfies either $(b^*)$ or $(c^*)$, which appear in Lemma 12.

Therefore, we can assume that there exists a $\xi_1 \in (0,1]$ such that

$$w_1(\xi_1) = w_1'(\xi_1) = 0 \quad \text{and} \quad w_1'(r) \le 0 \quad \text{on} \quad [0,\xi_1].$$

Multiple $(E)$ by $ru'$ and integrate it from 0 to $\xi_1$, we have

$$\int_0^{\xi_1} (r^{n-1}|u'|^{m-2}u')' ru' + \int_0^{\xi_1} [u^p + \lambda u^q] r^n u' = 0.$$

Integrating by parts and using the fact $u'(0) = 0$, we have

(4)
$$\begin{aligned}
&(\frac{m-n}{m}) \int_0^{\xi_1} r^{n-1}|u'|^m dr \\
&= (\frac{m-1}{m})\xi_1^n |u'(\xi_1)|^m + \frac{u^{p+1}(\xi_1)}{p+1}\xi_1^n - \frac{n}{p+1}\int_0^{\xi_1} r^{n-1}u^{p+1}dr \\
&\quad + \lambda \frac{v^{q+1}(\xi_1)}{q+1}\xi_1^n - \frac{\lambda n}{q+1}\int_0^{\xi_1} r^{n-1}u^{q+1}dr.
\end{aligned}$$

On the other hand, we have

(5)
$$\int_0^{\xi_1} r^{n-1}|u'|^m dr = u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1)\xi_1^{n-1} + \int_0^{\xi_1} r^{n-1}[u^{p+1} + \lambda u^{q+1}]dr.$$

By (4)+$\frac{n}{p+1}\times$ (5), we see that

$$\begin{aligned}
&\left\{\frac{n}{p+1} - \frac{n-m}{m}\right\} \int_0^{\xi_1} r^{n-1}|u'|^m dr + \lambda n \left\{\frac{1}{q+1} - \frac{1}{p+1}\right\} \int_0^{\xi_1} r^{n-1}u^{q+1}dr \\
&= (\frac{m-1}{m})\xi_1^n|u'(\xi_1)|^m + \frac{n}{p+1}u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1)\xi_1^{n-1} + \frac{u^{p+1}(\xi_1)}{p+1}\xi_1^n + \frac{\lambda u^{q+1}(\xi_1)}{q+1}\xi_1^n.
\end{aligned}$$

It follows from

$$t_1 \in (0,1), \quad u(r) \geq t_1 v(r) \quad \text{and} \quad |u'(r)| \geq t_1 |v'(r)| \quad \text{on} \quad [0,\xi_1] \subseteq [0,1]$$

that we have

$$|u'(r)|^m - t_1^{q+1}|v'(r)|^m \geq t_1^m |v'(r)|^m (1 - t_1^{q+1-m}) = 0 \quad \text{on} \quad [0,\xi_1],$$

$$u^{q+1}(r) - t_1^{q+1} v^{q+1}(r) \geq t_1^{q+1} v^{q+1}(r) - t_1^{q+1} v^{q+1}(r) = 0, \quad \text{on} \quad [0,\xi_1],$$

$$\begin{aligned}&u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) - t_1^{q+1} v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1)\\ &= v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1)[t_1^m - t_1^{q+1}] = 0,\end{aligned}$$

$$u^{p+1}(\xi_1) - t_1^{q+1} v^{p+1}(\xi_1) = t_1^{p+1} v^{p+1}(\xi_1)[1 - t_1^{q-p}] < 0,$$

and

$$u^{q+1}(\xi_1) - t_1^{q+1} v^{q+1}(\xi_1) = 0.$$

Therefore, we have

$$\begin{aligned}0 \leq \; & \left\{\frac{n}{p+1} - \frac{n-m}{m}\right\} \int_0^{\xi_1} r^{n-1}[|u'|^m - t_1^{q+1}|v'|^m]dr\\ &+\lambda n \left\{\frac{1}{q+1} - \frac{1}{p+1}\right\} \int_0^{\xi_1} r^{n-1}[u^{q+1} - t_1^{q+1} v^{q+1}]dr\\ &= (\tfrac{m-1}{m})\xi_1^n[|u'(\xi_1)|^m - t_1^{q+1}|v'(\xi_1)|^{q+1}]\\ &+\frac{n}{p+1}\xi_1^{n-1}[u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) - t_1^{q+1}v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1)]\\ &+\frac{u^{p+1}(\xi_1) - t_1^{q+1}v^{p+1}(\xi_1)}{p+1}\xi_1^n + \frac{\lambda[u^{q+1}(\xi_1) - t_1^{q+1}v^{q+1}(\xi_1)]}{q+1}\xi_1^n < 0,\end{aligned}$$

which gives a contradiction.

This implies that $u(r) \equiv v(r)$ on $[0,\xi_1]$, thus $u(r) \equiv v(r)$ on $[0,1]$. □

**Theorem 15.** *(2-nd Uniqueness theorem) Suppose that*

$(C_5)$ $\psi(r) = 1$ *on* $[0,1]$, $f(r,u) := u^p + \lambda u^q$ *on* $[0,1]\times[0,\infty)$ *and satisfies*

$$2 \leq m \quad \text{and} \quad m = q+1 < p+1.$$

*Then for any* $\lambda \geq 0$, *boundary value problem (BVP) has at most one positive solution in* $C^2(0,1)\cap C^1[0,1]$.

*Proof.* Suppose that $u$ and $v$ are two distinct positive solutions of $(BVP)$. Then it follows from Theorem 13 that there exist $t_i \in (0,1), \xi_i \in [0,1]$ for $i = 1,2$ such that each of the functions $w_1(r) = u(r) - t_1 v(r)$ and $w_2(r) = v(r) - t_2 u(r)$ satisfies either $(b^*)$ or $(c^*)$, which appear in Lemma 12.

Therefore, we can assume that there exists a $\xi_1 \in (0,1]$ such that

$$w_1(\xi_1) = w_1'(\xi_1) = 0 \qquad \text{and} \qquad w_1'(r) \leq 0 \quad \text{on} \quad [0,\xi_1].$$

Multiple $(E)$ by $ru'$ and integrate it from 0 to $\xi_1$, we have

$$\int_0^{\xi_1} (|u'|^{m-2}u')'ru' + \int_0^{\xi_1} [u^p + \lambda u^q]ru' = 0.$$

Integrating by parts and using the fact $u'(0) = 0$, we have

$$\begin{aligned} &(\frac{m-1}{m}) \int_0^{\xi_1} |u'|^m dr \\ &= (\frac{m-1}{m})\xi_1 |u'(\xi_1)|^m + \frac{u^{p+1}(\xi_1)}{p+1}\xi_1 - \frac{1}{p+1}\int_0^{\xi_1} u^{p+1} dr \\ &\quad + \lambda \frac{v^{q+1}(\xi_1)}{q+1}\xi_1 - \frac{\lambda}{q+1}\int_0^{\xi_1} u^{q+1} dr. \end{aligned} \tag{6}$$

On the other hand, we have

$$\int_0^{\xi_1} |u'|^m dr = u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) + \int_0^{\xi_1} [u^{p+1} + \lambda u^{q+1}] dr. \tag{7}$$

By (6)+$\frac{1}{p+1}\times$ (7), we see that

$$\begin{aligned} &\left\{\frac{1}{p+1} - \frac{1-m}{m}\right\} \int_0^{\xi_1} |u'|^m dr + \lambda \left\{\frac{1}{q+1} - \frac{1}{p+1}\right\} \int_0^{\xi_1} u^{q+1} dr \\ &= (\frac{m-1}{m})\xi_1 |u'(\xi_1)|^m + \frac{1}{p+1}u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) + \frac{u^{p+1}(\xi_1)}{p+1}\xi_1 \\ &\quad + \frac{\lambda u^{q+1}(\xi_1)}{q+1}\xi_1. \end{aligned}$$

It follows from

$$t_1 \in (0,1), \quad u(r) \geq t_1 v(r) \quad \text{and} \quad |u'(r)| \geq t_1 |v'(r)| \quad \text{on} \quad [0,\xi_1] \subseteq [0,1]$$

that we have

$$|u'(r)|^m - t_1^{q+1}|v'(r)|^m \geq t_1^m|v'(r)|^m(1 - t_1^{q+1-m}) = 0 \quad \text{on} \quad [0,\xi_1],$$

$$u^{q+1}(r) - t_1^{q+1}v^{q+1}(r) \geq t_1^{q+1}v^{q+1}(r) - t_1^{q+1}v^{q+1}(r) = 0, \quad \text{on} \quad [0,\xi_1],$$

$$\begin{aligned} &u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) - t_1^{q+1}v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1) \\ &= v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1)[t_1^m - t_1^{q+1}] = 0, \end{aligned}$$

$$u^{p+1}(\xi_1) - t_1^{q+1}v^{p+1}(\xi_1) = t_1^{p+1}v^{p+1}(\xi_1)[1 - t_1^{q-p}] < 0,$$

and

$$u^{q+1}(\xi_1) - t_1^{q+1}v^{q+1}(\xi_1) = 0.$$

Therefore, we have

$$\begin{aligned} 0 \leq \quad & \left\{\frac{1}{p+1} - \frac{1-m}{m}\right\}\int_0^{\xi_1}[|u'|^m - t_1^{q+1}|v'|^m]dr \\ & +\lambda\left\{\frac{1}{q+1} - \frac{1}{p+1}\right\}\int_0^{\xi_1}[u^{q+1} - t_1^{q+1}v^{q+1}]dr \\ & = (\frac{m-1}{m})\xi_1[|u'(\xi_1)|^m - t_1^{q+1}|v'(\xi_1)|^{q+1}] \\ & +\frac{1}{p+1}[u(\xi_1)|u'(\xi_1)|^{m-2}u'(\xi_1) - t_1^{q+1}v(\xi_1)|v'(\xi_1)|^{m-2}v'(\xi_1)] \\ & +\frac{u^{p+1}(\xi_1) - t_1^{q+1}v^{p+1}(\xi_1)}{p+1}\xi_1 + \frac{\lambda[u^{q+1}(\xi_1) - t_1^{q+1}v^{q+1}(\xi_1)]}{q+1}\xi_1 < 0, \end{aligned}$$

which gives a contradiction.

This implies that $u(r) \equiv v(r)$ on $[0,\xi_1]$, thus $u(r) \equiv v(r)$ on $[0,1]$. □

## References


[1] Adimurthi, S.L. Yadava, An elemently proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet problem, *Arch. Rational Mech. Anal.*, **127** (1994), 219-229.

[2] R.P. Agarwal, V. Lakshmikantham, *Uniqueness and Nonuniqueness for Ordinary Differential Equations*, World Scientific, Singapore (1993).

[3] R.P. Agarwal, F.H. Wong, S.L. Yu, On the existence and non-existence of positive entire solutions of elliptic and parabolic equations, *Functional Differential Equations*, **4** (1997), 13-26.

[4] A. Ambrosetti, J.G. Azorero, I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Functional Anal.*, **137** (1996), 219-242.

[5] H. Brezis, L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Analysis TMA*, **19** (1986), 55-64.

[6] R. Dalmasso, Uniqueness of positive solutions of nonlinear second-order equations, *Proc. Amer. Math. Soc.*, **123** (1995), 3417-3424.

[7] B. Gidas, W.M. Ni, L. Nirenberg, Symmetric and related properties via the maximum principle, *Comm. Math. Phys.*, **68** (1979), 209-243.

[8] M.C. Knaap, L.A. Peletier, Quasilinear elliptic equations with nearly critical growth, *Comm. Part. Diff. Eqns.*, **14** (1989), 1351-1383.

[9] M.K. Kwong, Uniqueness of positive solution of $\Delta u - u + u^p = 0$ in $\mathbf{R}$, *Arch. Rational Mech. Anal.*, **105** (1989), 243-266.

[10] M.K. Kwong, On the Kolodner-coffman method for the uniqueness problem of Emden Fowler BVP, *Z. Angew. math. Phys.*, **41** (1990), 79-104.

[11] M.K. Kwong, Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.*, **333** (1992), 339-363.

[12] W.C. Lian, F.H. Wong, Uniqueness of positive solutions for quasilinear singular boundary value problem, *Applied Math. Letters*, **9** (1996), 53-59.

[13] Z. Liqun, Uniqueness of positive solutions of $\Delta u + u + u^p = 0$ in a finite ball, *Comm. Part. Diff. Eqns.*, **17** (1992), 1141-1164.

[14] W.N. Ni, R.D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(r, u) = 0$, *Comm. Pure Appl. Math.*, **38** (1985), 67-108.

[15] Y. Naito, Uniqueness of positive solutions of quasilinear differential equations, *Differential and Integral Equations*, **8** (1995), 1813-1822.

[16] P.N. Srikanth, Uniqueness of solutions nonlinar Dirichlet problems, *Differential and Integral Equations*, **6** (1993), 663-670.

[17] F.H. Wong, S.L. Yu, Uniqueness of positive solutions for singular nonlinear boundary value problems, *Hiroshima Mathematical Journal*, **26** (1996), 233-243.

[18] F.H. Wong, Uniqueness of positive solutions for Sturm-Liouville boundary value problems, *Proc. Amer. Math. Soc.*, **126** (1998), 365-374.

[19] F.H. Wong, An application of Schauder's fixed point theorem with respect higher order BVPs, *Proc. Amer. Math. Soc.*, **126** (1999), 2389-2397.

[20] F.H. Wong, Existence of positive solutions for m-Laplacian boundary value problems, *J. Applied Math. Letters.*, **12** (1999), 11-17.